Question bank on Mathematical Methods of Physics
Semester-I

1. Objective and short answer type questions:

a. What do you mean by scalar field and vector field ? Give an example of each.

b. If for two vectors $\vec{a}$ and $\vec{b}$, $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$, find the angle between $\vec{a}$ and $\vec{b}$.

c. Find the projection of the vector $\hat{i}-2\hat{j}+\hat{k}$ on $4\hat{i}-4\hat{j}+7\hat{k}$.

d. Find a unit vector normal to the surface $x^2+3y^2+2z^2=6$ at P(2,0,1).

e. Prove that $\oint\!\!\oint_S \vec{r}.d\vec{s}=3V$, where V is the volume bounded by the closed surface S .

f. Define Gamma function. Is there any restriction on the value of n and why ?

g. Prove that $\Gamma(1/2) = \sqrt{\pi}$.

h. What do you mean by an error function? Explain.

i. State the condition under which Legendre's equation gives a polynomial solution.

j. Write down the Taylor series expansion for the function f(x+h,y+k) upto terms containing second order derivatives.

k. Prove that the area of a parallelogram with sides $\vec{A}$ and $\vec{B}$ is $|\vec{A}\,X\,\vec{B}|$.

l. Find a unit vector perpendicular to the plane of $\vec{A}=2\hat{i}-6\hat{j}-3\hat{k}$ and $\vec{B}=4\hat{i}+\hat{j}-\hat{k}$.

m. Find the projection of the vector $\hat{i}+\hat{j}+3\hat{k}$ on $\hat{i}-2\hat{j}+3\hat{k}$.

n. Prove that $\vec{A}.(\vec{A}\times\vec{B})=0$.

o. What do you mean by an orthogonal curvilinear coordinate system?

p. Prove that $\Gamma(\frac{1}{2})=\sqrt{\pi}$.

q. What do you mean by ordinary point and regular singular point of a linear homogeneous second order ordinary differential equation?

r. From the generating function of Legendre polynomial, $(1-2xt+t^2)^{-1/2}$ = , show that $P_l(-x) = -P_l(x)$.

s. Find the volume of a parallelepiped with sides, $\vec{A}=2\hat{i}-\hat{j}-3\hat{k}$, $\vec{B}=4\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{C}=2\hat{i}+2\hat{j}-\hat{k}$.

t. Prove that $(\vec{A}\times\vec{B})^2= A^2B^2-(\vec{A}.\vec{B})^2$.

u. What do you mean by an orthogonal curvilinear coordinate system?

v. Define ordinary and singular points of a linear second order differential equation.

Broad answer type questions:

1. Find the general solution of Bessel's equation, $x^2 \frac{d^2 y}{d x^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right)y = 0,$ when n is not an integer.

2. Find a solution to the Fourier equation for heat conduction in one dimension $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$, that satisfies the condition, u(0,t)=0, u(l,t)=0(t$\geq$0) and u(x,0)=l-x when $\frac{l}{2}$<x<1.

3. a. Verify the expansion of the vector triple product $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$.

   b. Prove that $\vec{\nabla}\times(\vec{\nabla}\phi)=0$ and $\vec{\nabla}.(\vec{\nabla}\times\vec{A})=0$.

   c. Evaluate $\nabla^2(\ln r)$.

4. $\int_0^1 \left(1-x^n\right)^{1/n} dx = \frac{1}{n}\frac{\left[\Gamma(1/n)\right]^2}{2\Gamma(2/n)}$

5. $\int_0^{\pi/2} \cos^{2n}\varphi \; \mathrm{d}\varphi = \frac{\pi}{2}\frac{1.3.5.----(2n-1)}{2.4.6.----2n}$.

6. Solve the equation $\frac{dy}{dx} + x\sin 2y = x^3 \cos^2 y.$

7. Show that cos(xcosθ) = $J_0(x) - 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta$ -----, where J_n(x) is a Bessel function with the generating function, exp $[\frac{1}{2}x\left(t-\frac{1}{t}\right)] = \sum_{t=-\infty}^{t=+\infty} t^n J_n$ (x).

8. By the use of matrix method, solve the equation

$$x+y+z=7$$
$$x+2y+3z=16$$
$$x+3y+4z=22$$

9. (i) If A is a square matrix, show that $A+A^T$ is a symmetric matrix.
   (ii) Verify whether the matrix defined as

$$A=\frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}$$ is orthogonal.

   (iii) Find the eigen values of the matrix

$$A=\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

10. If x and y are real, solve the equation
$$\frac{iy}{ix+1}-\frac{3y+4i}{3x+y}=0$$

11. Prove that, $\log\frac{a+ib}{a-ib}=2i\tan^{-1}\frac{b}{a}$.

12.(i) What is an analytic function? Obtain the polar form of Cauchy-Riemann equations.
   (ii) Show that $\int_{-\infty}^{+\infty}\frac{dx}{x^4+1}=\frac{\pi}{\sqrt{2}}$.

13.Show that the matrix A is unitary.
$$A=\begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$$

14.Solve the equations by Cramer's rule.

$$3x+y+2z=3$$
$$2x-3y-z=-3$$
$$x+2y+z=4$$

15.Express $(\vec{\nabla}\, X\vec{A})$ in orthogonal curvilinear coordinates.

16.(a) Evaluate: $\int_0^1 x^4[\ln(\frac{1}{x})]^3 dx$

(b) Show that, $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma(\frac{1}{n})}{\Gamma(\frac{1}{2}+\frac{1}{n})}$

17. Show that 1.3.5-----(2n-1) = $\frac{2^n}{\sqrt{\pi}}\Gamma(n+\frac{1}{2})$

18. (a) Show that, $\vec{\nabla} X (\vec{C} X \vec{r}) = 2\vec{C}$, where $\vec{C}$ is a constant vector.

(b) Apply green's theorem in a plane to evaluate the integral $\oint(2x-y^3)dx - xy\,dy$ over the boundary of the region bounded by the circles $x^2+y^2=1$ and $x^2+y^2=9$.

(c) Solve the differential equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ for the conduction of heat along a rod without radiation, subject to the following conditions: (i) u is not infinite for t$\rightarrow\infty$, (ii) $\frac{\partial u}{\partial x} = 0$ *for* x=0 and x=l, (iii) u=lx-x^2 for t=0, between x=0 and x=l.

(d) Show that the function f(x)= $\begin{cases} 0 \text{ for } -\pi \le x < 0 \\ x \text{ for } 0 \le x < \pi \end{cases}$ can be expanded in Fourier series as $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \sum_{n=1}^{\infty}(-1)^n \frac{\sin nx}{n}$, Hence, show that $1+\frac{1}{3^3}+\frac{1}{5^2}+---=\frac{\pi^2}{8}$.

(e) (i) Give a physical interpretation of the divergence of a vector point function.

(ii) Show that $\vec{\nabla}\phi$ is a vector perpendicular to the surface ϕ(x,y,z) = c, where c is a constant.

(iii) State and explain Gauss divergence theorem.

(f) (i) What do you mean by orthogonal curvilinear coordinate system ? Show that the cylindrical co-ordinate system is orthogonal.

(ii) Obtain an expression for the curl of a vector field in orthogonal coordinates.

Hence find curl $\vec{A}$ in spherical polar coordinates.

(iii) Show that , $\int_0^{\pi/2} \cos^{2n}\phi d\phi = \frac{\pi}{2}\frac{1.3.5---(2n-1)}{2.4.6.---(2n)}$

[Given, $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\phi \cos^{2n-1}\phi d\phi$]

(iv) Evaluate

$$\int_0^{\pi/2} \sqrt{\tan\theta d\theta}$$

(g) (a) A periodic function f(x) with period 2π is defined as f(x)=x^2 ($-\pi \le 0 \le \pi$). Determine the Fourier coefficients. Show the nature of the function graphically.

(b) Starting from the generating function for Legendre polynomials prove the following recurrence relations:

i. $(l+1)P_{l+1}(x) - (2l+1)xP_l(x) + lP_{l-1}(x) = 0$ and
ii. $lP_l(x) = xP'_l(x) - P'_{l-1}(x)$

(b) Prove that $\vec{V}(x,y,z) = \{2xyz\hat{i} + (x^2 z + 2y)\hat{j} + x^2 y\hat{k}\}$ is irrotational and find the scalar potential.

(h) Evaluate $\oint\{(x^2 - 2xy)dx + (x^2 y + 3)dy\}$ around the boundary of the region bounded by $y^2 = 8x$ and $x = 2$ using Green's theorem.

(i) Show that the equation $\frac{dI(a)}{da} = 2I(a)$ is satisfied by I(a) = $\int_0^{\infty} e^{-x^2 - \frac{a^2}{x^2}} dx.$

Given $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$, show that I(a) = $\frac{\sqrt{\pi}}{2} e^{-2a}$.

(j) Solve the differential equation $x^2 \frac{d^2 y}{dx^2} - (x^2 + 2x)\frac{dy}{dx} + (x+2)y = x^2 e^x$.