SEMESTER-VI

PHYSICS-DSE: CLASSICAL DYNAMICS

Small Amplitude Oscillations

Small Amplitude Oscillations: Minima of potential energy and points of stable equilibrium, expansion of the potential energy around minimum, small amplitude oscillations about the minimum, normal modes of oscillations example of N identical masses connected in a linear fashion to (N -1) - identical springs.

The theory of small oscillations was developed by D' Alembert, Joseph-Louis Lagrange and other scientists. The study of the effect of all possible small perturbations to a dynamical system in mechanical equilibrium is known as small oscillations. The theory of small oscillations is widely used in different branches of physics viz. in acoustics, molecular spectra, coupled electric circuits etc.

1. **Equilibrium and its types-**

   Equilibrium is the condition of a system in which all the forces- internal as well as external, cancel out for some configuration of the system and unless the system is perturbed by an external agency, it stays indefinitely in that state.

   Equilibrium may be of following types-

   a) Static Equilibrium-

      Static equilibrium is a state of zero kinetic energy that continues indefinitely. In such equilibrium the immediate surroundings of the system does not change with time. Example- an object (e.g. book) lying still on a surface (e.g. table).

   b) Dynamic Equilibrium-

      Dynamic equilibrium is defined as the state when no net force acting on the system which continues with zero kinetic energy. In such equilibrium the immediate surroundings of the system change with time such that it exerts a balancing force on the system. Example- the charge neutrality of atoms.

   c) Stable Equilibrium-

      In stable equilibrium, if a small displacement is given, the system tends to return to the original equilibrium state. Example- the bob of a simple pendulum in equilibrium state.

   d) Unstable Equilibrium-

      In instable equilibrium, if a small displacement is given, the system does not return to the original equilibrium state. Example- a large sized stone lying on the upper edge of a cliff.

   e) Metastable Equilibrium-

      In metastable equilibrium, the system can't return to the original equilibrium configuration, if displaced sufficiently; for smaller displacement, however, it can return. Example- a balloon that explodes above a certain gas pressure.

2. **Equilibrium state and stability of a system-**

In a conservative system the potential V is a function of position (q_i) only and the system is said to be equilibrium when the generalized forces Q_i acting on the system vanish.

Therefore, $$Q_i = -\left(\frac{\partial V}{\partial q_i}\right)_0 = 0 \text{ ………………………… (i)}$$

i.e. the potential energy has an extremum (maximum or minimum) at the equilibrium configuration of the system, $q_{01}, q_{02}, q_{03}, q_{04}, \dots \dots \dots \dots \dots, q_{0n}$. When V is a minimum at equilibrium, any small deviation from this position means an increase in V and consequently decrease in kinetic energy T i.e., velocity *v*, in a conservative system and the body ultimately comes to rest, indicating the small bounded motion about the V_{min}. Such equilibrium is known as the stable equilibrium. On the other hand, a small departure from V_{max} results in decrease in V and increase in kinetic energy and velocity indefinitely, which corresponds to unstable motion. Thus the position of V_{max} is the position of unstable equilibrium. Therefore oscillations always occur about a position of stable equilibrium i.e. about V_{min}.

Let V(max) be the potential energy function for a particle and also let the force F acting on the particle vanishes at x_0 i.e.

$$F = -\left(\frac{dV}{dx}\right)_{x_0} = 0 \text{……………………………. (ii)}$$

In that case x_0 is the point of equilibrium. To test the stability of the equilibrium we must examine $\frac{d^2V}{dx^2}$ at x_0. If the second derivative is positive, the equilibrium is stable. Thus in stable equilibrium,

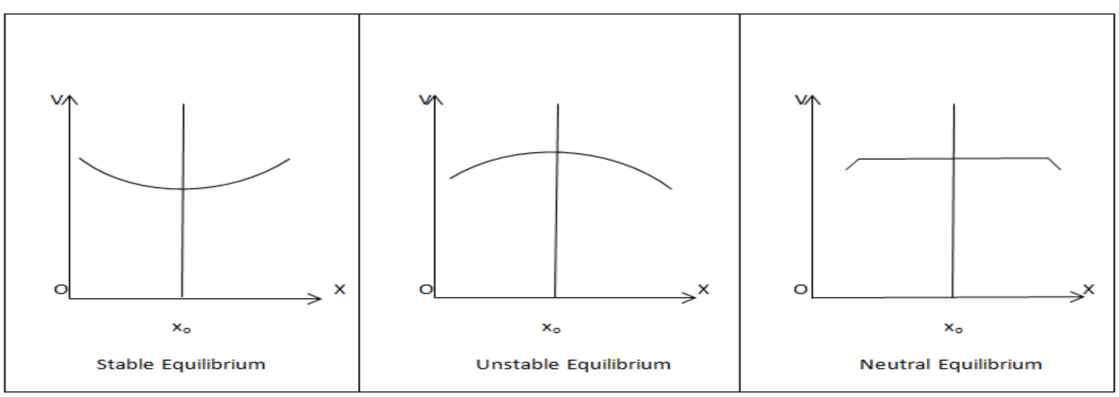
$$\left(\frac{d^2V}{dx^2}\right)_{x_0} > 0$$

If the second derivative is negative, the equilibrium is unstable. Thus in unstable equilibrium,

$$\left(\frac{d^2V}{dx^2}\right)_{x_0} < 0$$

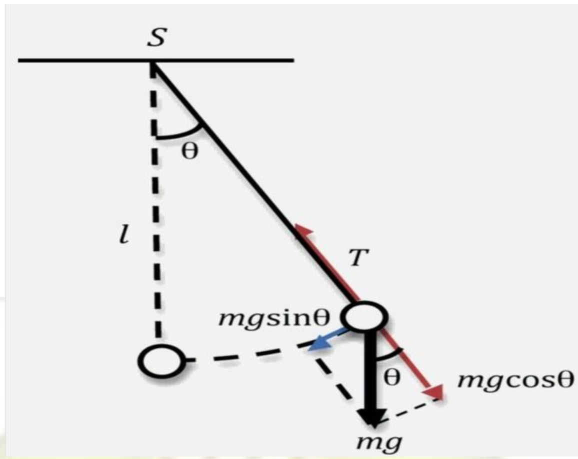
And if the second derivative also vanishes, we must examine the higher derivatives at x_0. If all the derivatives vanish at x_0, so that V(x) is constant in a region about x_0, then the particle is effectively free. In such a situation no force results from a displacement (from x_0) and the system to be in state of neutral stability. Thus in neutral equilibrium,

$$\left(\frac{d^2V}{dx^2}\right)_{x_0} = 0$$



3. **The Stability of a Simple Pendulum (One Dimensional Oscillator)-**

Let m be the mass of the bob of a pendulum and l be its length. The zero of the potential energy scale is taken at the bottom of the swing.



If Θ be its angular deflection, then the potential energy is given by,

$$V(\theta) = mgl(1 - Cos\theta) \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

where g is the acceleration due to gravity.

The equilibrium position of the pendulum is given by,

$$\frac{dV}{d\theta} = mgl\ Sin\theta = 0$$

$$i.e.\ \Theta = \ either\ 0\ or\ \pi$$

Now, we have

$$\frac{d^2V}{d\theta^2} = mgl\ Cos\theta$$

Now, $\left(\frac{d^2V}{d\theta^2}\right)_{\theta=0} > 0$, and it corresponds to the position of stable equilibrium.

$\left(\frac{d^2V}{d\theta^2}\right)_{\theta=\pi} < 0$, and it corresponds to the position of unstable equilibrium.

Therefore Θ=0 is the position of stable equilibrium at which the bob can hang for an indefinite period, which is not at all possible at the position of unstable equilibrium at $\theta = \pi$. The pendulum can therefore oscillate only about the position of stable equilibrium (Θ=0).

4. **General Problem of Small Oscillation**-

a) **Formulation of the Problem-**

In tackling the general problem of small oscillations we perform an appropriate simplification of Lagrangian method. We know that the oscillations whether large or small take place about a position of equilibrium. Thus a pendulum oscillates about a verticval which is its stable equilibrium position. When the pendulum is deflected from the stable position a restoring force acts on the pendulum in a direction opposite to the direction of deflection. In the equilibrium position this force is zero.

We now suppose that the force is acting on a system which is wholly conservative and hence it is derivable from potential. In the equilibrium condition of the system the generalized force Q_i will be equal to zero.

The potential energy is a function of position only i.e., the generalized co-ordinates $q_1, q_2, q_3, q_4, \ldots\ldots\ldots\ldots, q_n$. Hence,

$$\left(\frac{\partial V}{\partial q_i}\right)_0 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(i)$$

Let us denote the deviation of the generalized co-ordinates from equilibrium by η_i.

Therefore, $q_i = q_{0i} + \eta_i \; or, \eta_i = q_i - q_{0i}$……………………….......(ii)

q_{0i} denotes the equilibrium position of the co-ordinate system.

Expanding the potential V $(q_1, q_2, q_3, q_4, \ldots\ldots\ldots\ldots, q_n)$ in Taylor's series about q_{0i} we may write,

$$V\,(q_1, q_2, \ldots\ldots\ldots., q_n) = V(q_{01}, q_{02}, \ldots\ldots\ldots., q_{0n}) + \sum_i^n \left(\frac{\partial V}{\partial q_i}\right)_0 \eta_i + \frac{1}{2}\sum_{i,j}^{n,n}\sum \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i\,\eta_j + \cdots$$

………………….(iii)

The first term on the right hand side of the equation is constant and hence may be taken to be equal to zero and the second term vanishes due to equation (i). So the first approximation we may write,

$$V\,(q_1, q_2, \ldots\ldots\ldots., q_n) = \frac{1}{2}\sum_{i,j}^{n,n}\sum \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i\,\eta_j \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(iv)$$

Writing V_{ij} for $\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0$ we get,

$$V\,(q_1, q_2, \ldots\ldots\ldots., q_n) = \frac{1}{2}\sum_{i,j}\sum V_{ij}\,\eta_i\eta_j \ldots\ldots\ldots\ldots\ldots\ldots\ldots(v)$$

We shall now examine the expression for kinetic energy of the system of particles which is given by,

$$T = \sum_{i,j}\sum \frac{1}{2} m_{ij}\dot{q}_i\dot{q}_j$$

Here, $m_{ij} = m_{ji}, i.e. m_{ij}$ form a symmetric matrix. From Taylor's series we may write,

$$m_{ij}(q_1, q_2, \ldots\ldots\ldots., q_n) = m_{ji}(q_{01}, q_{02}, \ldots\ldots\ldots., q_{0n}) + \sum_k (\frac{\partial m_{ij}}{\partial q_k})_0\,\eta_k + \cdots$$

…………………………(vi)

Hence the kinetic energy becomes

$$T = \frac{1}{2}\{m_{ij}(q_{01}, q_{02}, \ldots\ldots\ldots., q_{0n}) + \sum_k \left(\frac{\partial m_{ij}}{\partial q_k}\right)_0 \eta_k + \cdots\}\dot{\eta}_i\dot{\eta}_j$$

This equation is quadratic in $\dot{\eta}_i$'s and the lowest non-vanishing approximation to T is obtained by dropping all the terms on the right hand side of the above equation except the first term.

Therefore,

$$T = \frac{1}{2}\sum\sum m_{ij}(q_{01}, q_{02}, \dots \dots \dots, q_{0n})\dot{\eta}_i\dot{\eta}_j = \frac{1}{2}\sum\sum T_{ij}\,\dot{\eta}_i\dot{\eta}_j$$

……………………………….(vii)

Here the constant value of m_{ij} at equilibrium is denoted by T_{ij}.

Hence the Lagrangian function is,

$$L = T - V = \frac{1}{2}\sum_{i,j}\sum(T_{ij}\,\dot{\eta}_i\dot{\eta}_j - V_{ij}\eta_i\eta_j)$$

Thus the n-equations of motion derived from the function L is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right) - \left(\frac{\partial L}{\partial \eta_i}\right) = 0 \quad (i = 1,2,3, \dots, n)$$

$$or, \sum T_{ij}\,\ddot{\eta}_j + \sum V_{ij}\eta_j = 0 \quad \text{…………………..(viii)}$$

In the above equation both T_{ij} and V_{ij} have symmetric properties.

We can write this equation in the following matrix form,

$$T\ddot{\eta} + V\eta = 0 \quad \text{…………….............……..(ix)}$$

where $T = \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \dots & \dots & \dots & \dots \\ T_{n1} & T_{n2} & \dots & T_{nn} \end{pmatrix}; V = \begin{pmatrix} V_{11} & V_{12} & \dots & V_{1n} \\ V_{21} & V_{22} & \dots & V_{2n} \\ \dots & \dots & \dots & \dots \\ V_{n1} & V_{n2} & \dots & V_{nn} \end{pmatrix}; \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}$

Here T denotes the inertial coefficient matrix and V the stiffness coefficient matrix. V plays the role of restoring force which tends to bring the system to the position of equilibrium which may be called restoring tensor.

b) **Eigen Value Problem and Normalization-**

For similarity to the equation of simple harmonic oscillator, we try an oscillator solution of equation (viii) in the form,

$$\eta_j = a_j e^{i\omega t} \quad \text{……………………………………(x)}$$

The idea behind the above trial solution is that ω is independent of *j*, i.e. all the co-ordinates are assumed to execute SHM with same period but different amplitude (and either in-phase or out-of-phase only, depending on the sign of a_j). a_j are in general complex and only the real parts of them correspond to the actual motion. Putting equation (x) in equation (viii) we get,

$$\sum_j (V_{ij} - \omega^2 T_{ij})a_j = 0$$

…………………..(xi)

$(i = 1,2,3, \dots \dots \dots \dots \dots, n)$

which is a set of n linear homogeneous equations for the a_j's and consequently for a non-trivial solution (i.e., not all a_j =0) the determinant of the coefficients must vanish i.e. $|V_{ij} - \omega^2 T_{ij}| = 0$

$$or, \begin{vmatrix} V_{11} - \omega^2 T_{11} & \dots & \dots & V_{1n} - \omega^2 T_{1n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ V_{n1} - \omega^2 T_{n1} & \dots & \dots & V_{nn} - \omega^2 T_{nn} \end{vmatrix} = 0$$

.... (xii)

This is called the secular determinant. This determinant condition is an algebraic equation of nth degree for ω^2 and hence provides n- values of ω^2 for which equation (x) represents the correct solutions to the equations of motion. For each of these values of ω^2 the equation (xi) may be solved for the amplitudes or more precisely for (n-1) of the amplitudes in terms of the remaining one. This is due to the fact that the coefficient a_j are determined only up to a common multiplier. This is apparent from equation (xi). Thus the ratios of a_j's are definite (for a particular frequency) but not their absolute values. This arbitrariness in the values of a_j will be utilized in the normalization procedure.

Since there exist n values of ω_k, we also get n- set of values of amplitudes. Each of these set (i.e., n- values of a_j's corresponding to a particular frequency(ω_k) can be considered to define the components of a n- dimensional vector $\vec{a}_k$. This $\vec{a}_k$ is called the eigen vector of the system associated with the eigen frequencyω_k. The symbol a_{jk} may now be used to represent the j-th component of the k-th eigen vector. Using the principle of superposition, the general solution may now be written as,

$$\eta_j(t) = \sum_k a_{jk} e^{i\omega_k t}$$

.... (xiii)

where motion of each η_j is compounded of motions with each of the n values of frequencies ω_k. The actual motion is the real part of the complex equation (xiii) which can be expressed as,

$$\eta_j(t) = \sum_k a_{jk}\, Cos(\,\omega_k t + \phi_k)$$

The motion is oscillatory about the stable equilibrium position only for $\omega_k^2 > 0$. If two or more values of ω_k happen to be the same, then the phenomenon is known as degeneracy.

c) **Orthogonality of Eigenvectors-**

Equation (xi) represents a special type of eigen value equation as,

$$\boldsymbol{Va} = \boldsymbol{\lambda Ta} \text{ ....(xiv) with } \omega^2 = \lambda$$

where the matrix V,T and a are

$$\boldsymbol{V} = \begin{pmatrix} V_{11} & V_{12} & \dots & V_{1n} \\ V_{21} & V_{22} & \dots & V_{2n} \\ \dots & \dots & \dots & \dots \\ V_{n1} & V_{n2} & \dots & V_{nn} \end{pmatrix}, \boldsymbol{T} = \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \dots & \dots & \dots & \dots \\ T_{n1} & T_{n2} & \dots & T_{nn} \end{pmatrix}, \boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

In ordinary eigenvalue problem, $\boldsymbol{Va} = \lambda \boldsymbol{a}$ i.e. the effect of V on eigenvector **a** is just to generate a vector which is λ times **a**, λ being a scalar. Here, however, **V** acting on **a**, generates a multiple (λ) of the result of T acting on **a**.

A typical equation for the eigenvalue λ_k can be written from equation (xiv) as,

$$\sum_j V_{ij} a_{jk} = \lambda_k \sum_j T_{ij} a_{jk}$$

.....................................(xv)

The complex conjugate of similar equation for λ_l gives,

$$\sum_j V_{ij} a_{il}^* = \lambda_l^* \sum_j T_{ij}\, a_{il}^*$$

....(xvi)

(since V_{ij} and T_{ij} are real and also symmetric)

Multiplying equation (xv) by a_{il}^* and summing over *i* and multiplying equation (xvi) by a_{jk} and summing over *j*, and next subtracting the latter from the former, we get

$$(\lambda_k - \lambda_l^*) \sum_{ij} T_{ij} a_{jk}\, a_{il}^* = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (xvii)$$

$$or, (\lambda_k - \lambda_k^*) \sum_{ij} T_{ij} a_{jk}\, a_{ik}^* = 0$$

....(xviii) for $l=k$

Now, $a_{jk} = \alpha_{jk} + i\beta_{jk}$ as a_{jk} is complex.

and

$$\sum T_{ij} a_{jk} a_{ik}^* = \sum_{ij} T_{ij} \alpha_{jk}\, \alpha_{ik} + \sum_{ij} T_{ij} \beta_{jk}\, \beta_{ik} + i \sum T_{ij} (\beta_{jk}\alpha_{ik} - \alpha_{jk}\beta_{ik}) \ldots\ldots\ldots\ldots.. (xix)$$

T_{ij} being symmetric, the interchange of dummy indices *i* and *j* changes sign of the third summation on the right. Therefore, the imaginary term vanishes in equation (xix) and the two real sums are twice the kinetic energy when the velocities $\dot{\eta}_i$ have values α_{ik} and β_{ik} respectively. The kinetic energy can never be zero for real velocities. Thus, the eigenvalues λ_k must be real.

Eigenvalues and eigenvectors being real, the equation (xvii) may be written as

$$(\lambda_k - \lambda_l) \sum_{ij} T_{ij} a_{jk}\, a_{il} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (xx)$$

When the roots of the characteristic equation (xii) are all distinct, $\lambda_k \neq \lambda_l$, we shall get,

$$\sum T_{ij} a_{jk}\, a_{il} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (xxi)(for\ l \neq k)$$

This shows that the eigenvectors are orthogonal.

The eigenvalue equation (xi) can't completely fix the values of a_{jk}, as has already been stated. The uncertainty can however be removed be removed by setting, purely arbitrarily,

$$\sum_{ij} T_{ij} a_{jk}\, a_{ik} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (xxii)(for\ l = k)$$

Each of n such equations uniquely fix the arbitrary component in each eigenvector (corresponding to a given eigenfrequency). Combining the equations (xxi) and (xxii), we get

$$\sum_{ij} T_{ij} a_{jk}\, a_{ik} = \delta_{lk}$$

$$i.e.\, \boldsymbol{\tilde{a}Ta} = \boldsymbol{I} \text{..............................(xxiii)}$$

where $\boldsymbol{\tilde{a}}$ being the transpose of **a**.

Introducing a diagonal matrix λ with elements $\lambda_{lk} = \lambda_k \delta_{lk}$, the eigenvalue equation (xv) becomes

$$\sum_j V_{ij} a_{jk} = \lambda_k \sum_j T_{ij} a_{jk} = \sum_{ji} T_{ij}\, a_{jl} \lambda_{lk}$$

$$i.e. \boldsymbol{Va} = \boldsymbol{Ta}\lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (xxiv)$$

Therefore, $\boldsymbol{\tilde{a}Va} = \boldsymbol{\tilde{a}Ta}\lambda = \lambda$.........................(xxv)

This shows that the matrix **a** diagonalises both **T** and **V** simultaneously.

d) **Normal co-ordinates-**

The normalization of a_{jk} leaves no ambiguity in the solution for η_j. To remove the loss of generality due to the introduction of arbitrary normalization, a scalar factor β_k may be used when the equation (xiii) becomes

$$\eta_j(t) = \sum_k a_{jk}\beta_k e^{i\omega_k t} = \sum_k a_{jk}Q_k \dots\dots\dots\dots\dots\dots\dots\dots (xxvi)$$

where $Q_k = \beta_k e^{i\omega_k t}$............................(xxvii)

Therefore, $\tilde{\boldsymbol{\eta}} = \tilde{\boldsymbol{a}}\tilde{\boldsymbol{Q}}$..........................(xxviii)

Equation (xxvii) shows that Q_k oscillates with one frequency only, called normal frequency, namely ω_k. These Q_k's are called the normal co-ordinates of the system. The normal co-ordinates are thus defined as the generalized co-ordinates where each one of them executes oscillations with a single frequency. If expressed in terms of normal co-ordinates, potential energy and kinetic energy take rather simple forms as given below,

$$P.E. = V = \frac{1}{2}\sum_{ij}\eta_i V_{ij}\eta_j = \frac{1}{2}\tilde{\boldsymbol{\eta}}\boldsymbol{V}\boldsymbol{\eta} = \frac{1}{2}\tilde{Q}\tilde{\boldsymbol{a}}\boldsymbol{V}\boldsymbol{a}Q = \frac{1}{2}\tilde{Q}\lambda Q$$

$$= \frac{1}{2}\sum_{lk} Q_l\lambda_{lk}Q_k = \frac{1}{2}\sum_{lk} Q_l\lambda_k\delta_{lk}Q_k = \frac{1}{2}\sum_k \omega_k^2 Q_k^2 \dots\dots\dots\dots..(xxix)$$

(using equations (xxviii) and (xxv))

$$K.E = T = \frac{1}{2}\sum_{ij}\dot{\eta}_i T_{ij}\dot{\eta}_j = \frac{1}{2}\tilde{\dot{\eta}}T\dot{\eta} = \frac{1}{2}\tilde{\dot{Q}}\tilde{\boldsymbol{a}}\boldsymbol{T}\boldsymbol{a}\dot{Q} = \frac{1}{2}\tilde{\dot{Q}}\dot{Q} = \frac{1}{2}\sum_k \dot{Q}_k^2 \dots\dots\dots\dots\dots\dots.(xxx)$$

So, the new Lagrangian is,

$$L = T - V = \frac{1}{2}\sum_k (\dot{Q}_k^2 - \omega_k^2 Q_k^2) \dots\dots\dots\dots\dots\dots\dots\dots..(xxxi)$$

Therefore, the Lagrange's equation of motion are given by,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_k}\right) - \frac{\partial L}{\partial Q} = 0$$

$$or, \ddot{Q}_k + \omega_k^2 Q_k = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (xxxii)$$ (k=1,2,3......)

This is the set of n-independent equations of SHM with single normal frequency ω_k.

The solution of the equation (xxxii) is given by,

$$Q_k = a_k Cos(\omega_k t) + b_k Sin(\omega_k t), when\ \omega_k^2 > 0$$

$$Q_k = a_k t + b_k t, when\ \omega_k^2 = 0$$

$$Q_k = a_k e^{\omega_k t} + b_k e^{-\omega_k t}, when\ \omega_k^2 < 0$$

For $\omega_k^2 > 0$, the co-ordinates always remain finite. Such solutions refer to stable equilibrium. For $\omega_k^2 = 0$ and $\omega_k^2 < 0$, the co-ordinates become infinite as time advances and such solutions refer to unstable equilibrium.

e) Normal co-ordinates Q_k in terms of co-ordinates η_j

The complex quantity β_k may be written as,

$$\beta_k = \mu_k + i\nu_k \text{...(xxxiii)}$$

Therefore, equation (x) becomes,

$$\eta_j(t) = \sum_k a_{jk}(\mu_k + i\nu_k)e^{i\omega_k t}$$

............................(xxxiv)

On differentiating with respect to time we get,

$$\dot{\eta}_j(t) = \sum_k i\omega_k a_{jk}(\mu_k + i\nu_k)e^{i\omega_k t}$$

....(xxxv)

The real part of equation (xxxiv) gives at t=0,

$$\eta_j(0) = \sum_k a_{jk}\,\mu_k$$

....(xxxvi)

Multiplying throughout by $T_{ji}a_{ir}$ and summing over ij, and using equation (xxxvi), we get,

$$\sum_{ij}\eta_j(0)T_{ji}a_{ir} = \sum_k \mu_k \sum_{ij} a_{jk}\,T_{ji}a_{ir} = \sum_k \mu_k\delta_{kr} = \mu_r$$

..........................(xxxvii)

Similarly, the real part of equation (xxxv) at t=0 gives,

$$\dot{\eta}_j(0) = -\sum_k \omega_k\,\nu_k a_{jk}$$

..................................(xxxviii)

Multiplying both sides by $T_{ji}a_{ir}$ and summing over ij, and using equation(xxxvi), we get,

$$\sum_{ij}\dot{\eta}_j(0)T_{ji}a_{ir} = -\sum_k \omega_k\nu_k \sum_{ij} a_{jk}\,T_{ji}a_{ir} = -\sum_k \omega_k\nu_k\delta_{kr} = -\omega_r\,\nu_r$$

.......................(xxxix)

Thus, the normal co-ordinates Q_r may be expressed as the real part of the expression in,

$$Q_r(t) = \beta_r e^{i\omega_r t} = (\mu_r + i\nu_r)e^{i\omega_r t} = \sum_{ij} T_{ji}a_{ir}[\,\eta_j(0) - \frac{i}{\omega_r}\dot{\eta}_j(0)]e^{i\omega_r t}$$

Therefore, for any arbitrary $\eta_j(0)$ and $\dot{\eta}_j(0)$, a set of normal co-ordinates Q_r may be found, each of which varies harmonically with single frequency ω_r.

f) Energy in normal co-ordinates

Here,

$V = \frac{1}{2}\sum_k \omega_k^2 Q_k^2$ and $T = \frac{1}{2}\sum_k \dot{Q}_k^2$

So, total mechanical energy E, in normal co-ordinates, is given by,

$$E = T + V = \frac{1}{2}\sum_k(\dot{Q}_k^2 + \omega_k^2 Q_k^2)$$

This is true for small oscillations with any number n of degrees of freedom.

Examples-

1. The potential energy V(x) of a particle is given by,

$$V(x) = 3x^4 - 8x^3 - 6x^2 + 24x$$

Determine the points of stable and unstable equilibrium.

Solution-

For the points of equilibrium$(dV/dx) = 0$. Since, we have

$$V(x) = 3x^4 - 8x^3 - 6x^2 + 24x$$

So, $\frac{dV}{dx} = 12x^3 - 24x^2 - 12x + 24$

At the points of equilibrium, therefore, $12x^3 - 24x^2 - 12x + 24 = 0$

$$or, x^3 - 2x^2 - x + 2 = 0$$

$$or, (x+1)(x-1)(x-2) = 0$$

Therefore, the points of equilibrium are

x= -1,1 and 2.

To test the stability of equilibrium at those points, we find $\frac{d^2V}{dx^2}$ at the points and if it is positive, the equilibrium is stable; if negative, it is unstable.

Now, $\frac{d^2V}{dx^2} = 36x^2 - 48x - 12$

So, $\left(\frac{d^2V}{dx^2}\right)_{x=-1} = 72\ (positive)$

$$\left(\frac{d^2V}{dx^2}\right)_{x=1} = -24\ (negative)$$

$$\left(\frac{d^2V}{dx^2}\right)_{x=2} = 36\ (positive)$$

Therefore, equilibrium are stable at x = -1 and x = 2; unstable at x = 1.

2. Consider two identical simple harmonic oscillators coupled together. Their K.E. and P.E. are given by $T = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2)$ and $V = \frac{1}{2}k(q_1^2 + q_2^2) - hq_1q_2$
   - (i) Find the Hamiltonian of the system.
   - (ii) Fine the frequencies for normal modes and the ratio of the amplitudes of each of the coordinates q_1 and q_2 in those modes.
   - (iii) Introduce the normal coordinates and thereby set up the corresponding Lagrange's equation.

   Solution-

   (i) Here, V is independent of the velocity, so it represents a conservative system. Thus the Hamiltonian is given by,

   $H = T + V = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2}k(q_1^2 + q_2^2) - hq_1q_2$

   (ii) Here, Lagrangian is

   $L = T - V = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}k(q_1^2 + q_2^2) + hq_1q_2$....................(i)

   Now, we get,

$$\frac{\partial L}{\partial \dot{q}_1} = m\dot{q}_1$$

$$-\frac{\partial L}{\partial q_1} = kq_1 - hq_2$$

$$\frac{\partial L}{\partial \dot{q}_2} = m\dot{q}_2$$

$$-\frac{\partial L}{\partial q_2} = kq_2 - hq_1$$

Putting these values in equation (i) we get,

$m\ddot{q}_1 + kq_1 - hq_2 = 0$..................(ii)

$m\ddot{q}_2 + kq_1 - hq_2 = 0$..................(iii)

Let $q_1 = A_1 e^{i\omega t}$. Putting this value in equation (ii) we get,

$$A_1(k - \omega^2 m) - A_2 h = 0$$

Again, let $q_2 = A_2 e^{i\omega t}$. Putting this value in equation (iii) we get,

$$-A_1 h + A_2(k - \omega^2 m) = 0$$

For non-zero solution, we must have the determinant,

$$\begin{vmatrix} k - \omega^2 m & -h \\ -h & k - \omega^2 m \end{vmatrix} = 0$$

$$or, (k - \omega^2 m)^2 - h^2 = 0$$

$$or, k - \omega^2 m = \pm h$$

Therefore, $\omega^2 = (k \mp h)$

So, $\omega_1 = \sqrt{\frac{(k-h)}{m}}, \omega_2 = \sqrt{\frac{(k+h)}{m}}$

Now, $\frac{A_2}{A_1} = \frac{(k-\omega_1^2 m)}{h}\ or, \frac{(k-\omega_2^2 m)}{h}$

(iii) Let the normal coordinates are $Q_i = A_i e^{i\omega_i t}$

Differentiating with respect to t we get,

$$\ddot{Q}_i + \omega_i^2 Q_i = 0, i = 1,2.$$

Thus Lagrange's equation is,

$$L_i = \frac{1}{2}\dot{Q}_i^2 - \frac{1}{2}\omega_i Q_i^2$$

Exercise-

1. The potential energy of a particle is given by
$$V(x) = x^4 - 4x^3 - 8x^2 + 48x$$
Find the points of stable and unstable equilibrium. (KNU-2019)
2. Discuss the general theory of small oscillation of a system to obtain the equation of motion near its equilibrium position. How the eigen frequencies of the system can be obtained for such a system. (KNU-2019)
3. What is normal mode and normal frequency? (KNU-2019)
4. Discuss the stability of a simple pendulum and show that it can oscillate only about the position of its stable equilibrium.
5. Establish the relation
$T\ddot{\eta} + V\eta = 0$,for small oscillations of a system, the symbols having their usual meanings.
6. A particle moves in potential energy given by

$V(x) = bx^2 + \frac{a}{x^2}$, a,b>0. Show that its frequency of oscillation is $\sqrt{\frac{8b}{m}}$.

7. What is meant by equilibrium, stable and unstable equilibrium? Show that oscillations can occur only about a stable equilibrium.
8. Write down the differences between stable and unstable equilibrium.

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