Lecture Notes

on

Sequence of Real Numbers

Course Code : CC-3

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Contents:

Sequence , Bounded sequence , Convergent sequence , Limit of a sequence , limit infemum , limit supremum , Limit theorems , Monotone sequence , Monotone convergence theorem , Subsequences , Divergence Criteria , Monotone subsequence theorem (statement only) , Bolzano Weiertrass theorem for sequence , Cauchy's convergence criterion.

Lecture 1 : Introduction to real sequences , Limit of a sequence .

Lecture 2 : Limit of sequence(continued) , Algebra of limits .

Lecture 3 : Sandwich Theorem , Divergent sequence , Some examples .

Lecture 4 : Monotone Sequence , Monotone Convergence Theorem, Subsequence .

Lecture 5 : Monotone subsequence theorem , Bolzano-Weiertrass theorem . Limit superior and limit inferior .

Lecture 6 : Subsequential limits , Cauchy's convergence criterion .

Section 1 : Introduction .

Suppose that for each positive integer n , we can find , by some determinable manner , a well-defined real number x_n . Then the collection { x_1 , x_2 , x_3 ,… x_n, ….} ,denoted also by $\{x_n\}$, is called a sequence of real numbers . More precisely we can define sequence as follows :

Definition :

A **sequence of real numbers** is a function from the set N of natural numbers to the set R of real numbers i.e. $\boldsymbol{f}: \boldsymbol{N} \rightarrow \boldsymbol{R}$ **.** If *f(n)* = x_n , then the sequence generated by f is denoted by *{f(1) ,f(2) , f(3),…….}* or *{ f(n)}* or $\{x_n\}$. *f(n)* is the n-th term of the sequence . Here we use the sequence to mean 'real sequence'.

Remark :

(a) It is to be born in mind that **a sequence $\{x_n\}$ is different from the set $\{x_n : n \in N\}$** . For instance , a number may be repeated in a sequence $\{x_n\}$,but it need not to be written repeatedly in the set . For example { 1,-1 ,1 ,-1 ,1 ,……….} is a sequence $\{x_n\}$ where $x_n = (-1)^n$ where $n \in \boldsymbol{N}$ **,** but where as the set $\{x_n : n \in \boldsymbol{N}\}$ is same as $\{1 , -1\}$.

(b) Instead of sequence of real numbers, we can also talk about a sequence of elements from any nonempty set S, such as sequence of sets, sequence of functions and so on. Thus, given a nonempty set S, a sequence in S is a function : $\boldsymbol{N} \rightarrow \boldsymbol{S}$ **.** For example ,for each $n \in \boldsymbol{N}$ consider the set $A_n = \{i \in \boldsymbol{N} : i \leq n\}$. Then we obtain a sequence of subsets of $\boldsymbol{N}$, namely $\{ A_1, A_2, A_3, \ldots \ldots \}$.

(c) The range of a sequence $\{x_n\}$ is a subset of $\boldsymbol{R}$, denoted by the symbol $\{x_n : n \in \boldsymbol{N}\}$.

Examples : (1) Let $f: \boldsymbol{N} \rightarrow \boldsymbol{R}$ be defined by *f(n)* = n^2 , $n \in N$. Then sequence is $\{n^2\}$. It is also denoted by $\{1^2,\ 2^2, 3^2, 4^2, \ldots \ldots\}$. Range of the sequence is also $\{1^2,\ 2^2, 3^2, 4^2, \ldots \ldots\}$.

(2) Let $f: \boldsymbol{N} \rightarrow \boldsymbol{R}$ be defined by *f(n)* = $(-1)^n$, $n \in N$. The sequence is $\{(-1)^n\}$ .It is also denoted by $\{1, -1, 1, -1, \ldots \ldots\}$ .The range of the sequence is $\{1, -1\}$.

(3) Let $f: \boldsymbol{N} \rightarrow \boldsymbol{R}$ be defined by *f(n)* = 2 , $n \in N$. The sequence is $\{2,2,2,2, \ldots \ldots\}$ .It is called constant sequence . Range of the sequence is $\{2\}$.

(4) The celebrated **Finonacci sequence** $\{x_n\}$ is given by the inductive definition

$$x_1 = 1 , \qquad x_2 = 2 , \qquad x_n = x_{(n-1)} + x_{(n-2)} , n \geq 3$$

Thus each term is the sum of its two immediate predecessors. The sequence may be written as $\{1, 1, 2, 3, 5, 8, 13, 21 , \ldots \ldots \ldots \ldots\}$.

Geometrical Representation :

Each term of the sequence $\{x_n\}$ of real numbers corresponds to a point on the real axis. We then call the collection of such points a sequence of points on the real line.
On the plane determined by two mutually perpendicular lines OX and OY , we may plot the points (n , x_n) and thus obtain a sequence of points on the plane. OX is the axis representing n and OY is the axis representing x_n.

Bounded Sequence :

As we know about the bounds of a set of real numbers , the similar definition is applicable of sequence also.

A real sequence $\{x_n\}$ is said to be ***bounded above*** if there exists a real number **G** such that $\boldsymbol{x_n} \leq$ **G** for all n ∈ N . **G** is said to be upper bound of the sequence.

A real sequence $\{x_n\}$ is said to be ***bounded below*** if there exists a real number **g** such that $\boldsymbol{x_n} \geq$ **g** for all n ∈ N . **g** is said to be lower bound of the sequence.

A real sequence $\{x_n\}$ is called **bounded sequence** if it is bounded above as well as bounded below. In that case the range of the sequence is a bounded set.

Note :

If a sequence $\{x_n\}$ is bounded above then the range set is also bounded above and by the supremum property of **R ,** it has the least upper bound M , which is called the **supremum or least upper bound** of the sequence $\{x_n\}$. It satisfies the following properties :

(i) $x_n \leq M$ for all $n \in N$

(ii) for each pre-assigned positive ϵ ,there exists a $k \in N$ such that $x_k > M - \epsilon$.

By similar arguments ,if a sequence $\{x_n\}$ is bounded below , then its has the **infemum or greatest lower bound** *m* ,which satisfies the following condition :

(i) $x_n \geq m$ for all $n \in N$

(ii) for each pre-assigned positive ϵ ,there exists a $k \in N$ such that $x_k < m + \epsilon$.

For a unbounded above sequence $\{x_n\}$, we define M = sup$\{x_n\} = \infty$ and for a unbounded below sequence $\{y_n\}$,we define m = inf $\{y_n\} = -\infty$.

Examples : 1) The sequence $\left\{\frac{3n+1}{n+2}\right\}$ bounded. It can shown as follows :

$$0 \leq \frac{3n+1}{n+2} = \frac{3n+6-5}{n+2} = \frac{3(n+2)-5}{n+2} = 3 - \frac{5}{n+2} \leq 3 \quad \text{for all } n \in N .$$

Therefore , 3 is the least upper bound and 0 is the greatest lower bound.

2) The sequence $\{\frac{1}{n}\}$ is a bounded sequence . 0 is the greatest lower bound and 1 is the least upper bound of the sequence.

3) The sequence $\{n^2\}$ is bounded below and unbounded above . Here $\sup\{n^2\} = \infty$ and $\inf\{n^2\} = 1$.

4) Consider the sequence $\{(-1)^n n\}$ i.e $\{-1, 2, -3, 4, \dots \dots \dots .\}$. The sequence is both unbounded above and unbounded below . Here $\sup\{(-1)^n n\} = \infty$ and $\inf\{(-1)^n n\} = -\infty$.

Section 2 : Limit of a sequence .

A real sequence $\{x_n\}$ is said to be **convergent** to a real number l or l is said to be **limit of the sequence** $\{x_n\}$, if for a pre-assigned positive number $\epsilon > 0$ there exists a natural number k (depending on ϵ) such that , $|x_n - l| < \epsilon$ for all $n \geq k$ i.e. $l - \epsilon < x_n < l + \epsilon$ for all $n \geq k$ i.e. $x_n \in (l - \epsilon, l + \epsilon)$ for all $n \in N$. In other words for $n \geq k$, x_n belongs to the ϵ- neighbourhood of l.

If a sequence has a limit, we say that the sequence is **convergent**; if it has no limit, we say that the sequence is **divergent**. When a sequence has a limit , we will use the notation

$$\lim_{n\to\infty} x_n = l \text{ or } \lim x_n = l$$

We will sometimes use the symbolism $x_n \to l$, which indicates the intuitive idea that the values x_n ''approach'' the number l as $n \to \infty$.

Note: In order to establish the limit l of a sequence $\{x_n\}$, we are to start with an arbitrary positive number ϵ and then find some positive integer k such that $|x_n - l| < \epsilon$ for all $n \geq k$. Note that , different ϵ can result different k i.e. the number k may vary as ϵ varies.

Theorem 2.1 : (Uniqueness of limit) A sequence can have at most one limit .

Proof : If possible let l_1 and l_2 are two distinct limits of the sequence $\{x_n\}$,where $l_1 < l_2$.

Let $= (l_1 - l_2)/2$. Then $\epsilon > 0$ and $l_1 + \epsilon = l_2 + \epsilon$. Therefore the neighbourhoods $(l_1 - \epsilon, l_1 + \epsilon) and (l_2 - \epsilon, l_2 + \epsilon)$ are disjoint . Since l_1 and l_2 are both limits of the sequence ,for the chosen ϵ , there exists natural number k_1 and k_2 such that ,

$$l_1 - \epsilon < x_n < l_1 + \epsilon \text{ ,for all } n \geq k_1 \text{ and } l_2 - \epsilon < x_n < l_2 + \epsilon \text{ ,for all } n \geq k_2 .$$

Let $k = \max\{k_1, k_2\}$. Then ,

$$l_1 - \epsilon < x_n < l_1 + \epsilon \text{ and } l_2 - \epsilon < x_n < l_2 + \epsilon \text{ ,for all } n \geq k .$$

Therefore , $x_n \in (l_1 - \epsilon, l_1 + \epsilon) \cap (l_2 - \epsilon, l_2 + \epsilon)$,for all $n \geq k$. But this can not happen since the neighbourhoods $(l_1 - \epsilon, l_1 + \epsilon)$ and $(l_2 - \epsilon, l_2 + \epsilon)$ are disjoint . So our assumption that $l_1 \neq l_2$ is wrong .

Hence $l_1 = l_2$ and this proves that limit of a sequence is unique . ■

Alternative Proof : Note that ,

$$|l_1 - l_2| = |l_1 - x_n + x_n - l_2| \leq |x_n - l_1| + |x_n - l_2|$$

Since the sequence $\{x_n\}$ converges to l_1 and l_2 , for a pre-assigned positive $\epsilon > 0$, there exists two natural number k_1 and k_2 such that ,

$$|x_n - l_1| < \epsilon\backslash 2 \text{ for all } n \geq k_1 \quad \text{and} \quad |x_n - l_2| < \epsilon\backslash 2 \text{ for all } n \geq k_2$$

Let $k = \max\{k_1 , k_2\}$. Then it follows that ,

$$|l_1 - l_2| \leq |x_n - l_1| + |x_n - l_2| < \epsilon \quad \text{for all } n \geq k .$$

Since this is true for every > 0 , it follows that $l_1 = l_2$. ■

Remark : Suppose $\{x_n\}$ is a sequence and $l \in R$. To show that $\{x_n\}$ does not converges to l ,we should be able to find an ϵ such that infinitely many point of $\{x_n\}$ are outside the interval $(l - \epsilon, l + \epsilon)$.

Exercise 2.1: Show that a sequence $\{x_n\}$ converges to l if and only if for every open interval ***I*** containing , there exists a natural number k such that $x_n \in I$ for all $n \geq k$.

Note : For $l \in R$ and $\epsilon > 0$, recall that the neighborhood of l is the set ,

$$N_\epsilon(l) = \{x \in R : |x - l| < \epsilon\} .$$

Since $x \in N_\epsilon(l)$ is equivalent to $|x - l| < \epsilon$, the definition of convergence of a sequence can be formulated in terms of neighborhoods. We give several different ways of saying that a sequence $\{x_n\}$ converges to l in the following theorem.

Theorem 2.2 : Let $\{x_n\}$ is a sequence of real number and let $l \in \boldsymbol{R}$ **.** Then the **following statements are equivalent :**

(a) $\{x_n\}$ is convergent to l .

(b) For every > 0 , there exists a natural number k such that $n \geq k$, the term x_n satisfy $|x_n - l| < \epsilon$.

(c) For every > 0 , there exists a natural number k such that $n \geq k$, the term x_n satisfy $l - \epsilon < x_n < l + \epsilon$.

(d) For every neighborhood $N_\epsilon(l)$ of l ,there exists a natural number k such that for all $n \geq k$, the terms x_n belongs to $N_\epsilon(l)$.

Proof : The equivalence of (a) and (b) is just the definition. The equivalence of (b) ,(c) and (d) follows from the implications :

$$|x - l| < \epsilon \iff -\epsilon < x - l < \epsilon \iff l - \epsilon < x < l + \epsilon \iff x \in N_\epsilon(l) . \quad ■$$

■ With the idea of neighborhoods one can describe convergence of the sequence $\{x_n\}$ to a real number l as : for each ϵ-neighborhood $N_\epsilon(l)$ of l , all but a finite number of terms of $\{x_n\}$ belongs to $N_\epsilon(l)$.

Example 2.1:

(1) The sequence $\left\{\frac{1}{n}\right\}$ is convergent to 0 .

(2) The sequence $\left\{\frac{2n+1}{n+1}\right\}$ is convergent to 2.

(3) Every constant sequence is convergent to the constant term in the sequence.

Solution :

(1) Let x_n =1/n for all $n \in N$ and let $\epsilon > 0$ be given . We have to identify a natural number k such that $\frac{1}{n} < \epsilon$ for all $n \geq k$. Note that ,

$$1/n < \epsilon \iff n > 1/\epsilon$$

Thus if we take $k = \left[\frac{1}{\epsilon}\right] + 1$,then we have

$$|x_n - 0| = \frac{1}{n} < \epsilon \quad \text{for all } n \geq k \text{ .}$$

Hence sequence $\left\{\frac{1}{n}\right\}$ is convergent to 0 .

(2) Let x_n =(2n+1)/(n+1) for all $n \in N$ and let $\epsilon > 0$ be given. We have to identify a natural number k such that ,

$$\left|\frac{2n+1}{n+1} - 2\right| < \epsilon \qquad \text{for all } n \geq k$$

i.e. $\quad 1/(n+1) < \epsilon \qquad$ for all $n \geq k$

i.e. $\quad n + 1 > 1\backslash\epsilon \qquad$ for all $n \geq k$

i.e. $\quad n > 1\backslash\epsilon - 1 \qquad$ for all $n \geq k$

Now if we take $= \left[\frac{1}{\epsilon} - 1\right] + 1$, then we have

$$|x_n - 2| = 1\backslash(n + 1) < \epsilon \text{ for all } n \geq k \text{ .}$$

Hence the sequence $\left\{\frac{2n+1}{n+1}\right\}$ is convergent to 2 .

(3) Let x_n = a for all $\in N$. The sequence is $\{a, a , a , a, \ldots \ldots \ldots .\}$. Let us choose $\epsilon > 0$. Now we have ,

$$|x_n - a| = 0 < \epsilon \quad \text{for all } n \geq 1 \text{ .}$$

Thus the sequence $\{x_n\}$ converges to a .

Definition : A sequence $\{x_n\}$ is called **eventually constant** if there exists $k \in N$ such that $x_{k+n} = x_k$ for all $n \geq 1$. The sequence is $\{x_1 , x_2 , \ldots \ldots , x_k , x_k , x_k , \ldots \ldots \ldots\}$.

Exercise 2.2 : Show that every eventually constant sequence converges .

Theorem 2.3: A convergent sequence is a bounded sequence .

Proof : Let the sequence $\{x_n\}$ converges to l . Since the sequence converges ,we can take any $\epsilon > 0$ we wish and so let us choose $\epsilon = 1$. For this ϵ there exists a natural number $k \in \boldsymbol{N}$ such that ,

$$|x_n - l\,| < \epsilon = 1 \quad \text{for all } n \geq k \text{ .}$$

Then we have that,

$$|x_n| = |x_n - l + l| \leq |x_n - l\,| + |l\,| < 1 + |\,l\,| = P \qquad \text{,for all } n \geq k \text{ .}$$

where $1 + |\,l\,| = P \in \boldsymbol{R}$.

Now define , $M = \max\{\,|x_1|\,, |x_2|\,, \ldots \ldots \ldots, |x_{k-1}|\,, P\}$ .Then $|x_n| \leq M$,for all $n \in N$. This proves that the sequence $\{x_n\}$ is bounded .

Remark : An unbounded sequence is not convergent . Also a bounded sequence may not be convergent sequence . For example $\{(-1)^n\}$ is a bounded sequence but the sequence does not converge to a limit .

Section 3 : The Algebra of Convergent sequence .

Theorem 3.1 : (Limit theorem) Let $\{x_n\}$ and $\{y_n\}$ are two convergent sequence that converges to *x* and *y* . Then ,

(i) lim $(x_n \pm y_n) = x \pm y$;

(ii) if $\in \boldsymbol{R}$ **,** lim $(cx_n) = cx$;

(iii) lim $x_n . y_n = x . y$;

(iv) lim $\frac{x_n}{y_n} = \frac{x}{y}$, provided $\{y_n\}$ is a sequence of non zero real numbers and $y \neq 0$.

Proof : Proof is left as a exercise . One can find the proof in any standard text book . ■

Theorem 3.2 : (Absolutely convergence) Let $\{x_n\}$ be a convergent sequence of real numbers converging to *x* . Then the sequence $\{|x_n|\}$ converges to $|x|$ i.e. every convergent sequence is absolutely convergent .

Proof : Let $\epsilon > 0$ be any given positive number . Since the sequence $\{x_n\}$ converges to *x* , there a natural number $k \in N$ such that , $|x_n - x| < \epsilon$ for all $n \geq k$.

Now we have ,

$$||x_n| - |x|| \leq |x_n - x| < \epsilon \quad \text{,for all } n \geq k .$$

Since ϵ is arbitrary , this proves that the sequence $\{|x_n|\}$ converges to $|x|$. ■

Note : The converse of the above theorem is not true . For example ,let $x_n = (-1)^n$ .Then the sequence $\{|x_n|\}$, which is a constant sequence $\{1,1,1,\ldots\ldots\}$, is converges to 1 but the sequence $\{x_n\}$ is not convergent .

Theorem 3.3 : If $\{x_n\}$ is a convergent sequence of real numbers and $x_n > 0$ for $n \in N$ .Then $x = \lim x_n \geq 0$.

Proof : Suppose the conclusion is not true and if possible let $x < 0$. Then $-x > 0$,a positive number . Let us choose a positive ϵ such that $0 < \epsilon < -x$ .Since $\{x_n\}$ converges to x ,there exists a natural number k such that ,

$$x - \epsilon < x_n < x + \epsilon \quad \text{for all } n \geq k .$$

In particular we have $x_k < x + \epsilon < 0$. But this contradicts the hypothesis that $x_n > 0$ for $\in N$. Therefore our assumption is not valid , this implies that $x \geq 0$. ■

Note : The above theorem is also true for a sequence $\{x_n\}$ such that $x_n \geq 0$ for $n \in N$.

Remark : If $\{x_n\}$ is a sequence of real numbers converges to x and suppose that $x_n \geq 0$ for all $n \in N$ .Then the sequence $\{\sqrt{x_n}\}$ of positive square roots converges and $\lim \sqrt{x_n} = \sqrt{x}$. We now give a useful result which is formally stronger than the previous theorem.

Theorem 3.4 : If $\{x_n\}$ and $\{y_n\}$ are two convergent sequence of real numbers and $x_n < y_n$ for all $n \in N$, then $\lim x_n \leq \lim y_n$.

Proof : Let $z_n = y_n - x_n$. Then by the given hypothesis $z_n > 0$ for all $\in N$. It follows from the previous theorem that ,

$$0 \leq \lim z_n = \lim y_n - \lim x_n ,$$

Consequently , $\lim x_n \leq \lim y_n$. ■

Note : The above theorem is also true for two convergent sequence $\{x_n\}$ and $\{y_n\}$ such that $x_n \leq y_n$ for all $n \in N$.

Exercise 3.1 : If $\{x_n\}$ is a convergent sequence real numbers and if $a \leq x_n \leq b$ for all $n \in N$, then $a \leq \lim x_n \leq b$.

Example 3.1: Let $x_n = \frac{1}{n+1}$ and $y_n = \frac{1}{n}$. Then $x_n < y_n$ for all $n \in N$ but ,

$$y_n - x_n = \frac{1}{n(n+1)} \to 0 \quad \text{, hence } \lim x_n = \lim y_n .$$

Theorem 3.5 (Sandwich Theorem) : Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequence of real numbers such that ,

$$x_n < y_n < z_n \quad \text{for all } n \in N$$

If $\lim x_n = \lim z_n = l$, then $\{y_n\}$ is convergent and $\lim y_n = l$.

Proof : Let $\epsilon > 0$. It follows from the convergence of the sequence $\{x_n\}$ and $\{z_n\}$ that there exists natural numbers k_1 and k_2 such that ,

$$|x_n - l| < \epsilon \quad \text{for all } n \geq k_1 \quad \text{and} \quad |z_n - l| < \epsilon \quad \text{for all } n \geq k_2 .$$

Let $= \max\{ k_1, k_2\}$. Then we have ,

$$l - \epsilon < x_n < l + \epsilon \quad \text{and} \quad l - \epsilon < z_n < l + \epsilon \quad \text{for all } n \geq k .$$

Therefore the given hypothesis implies that ,

$$l - \epsilon < x_n < y_n < z_n < l + \epsilon \quad \text{for all } n \geq k .$$

Consequently, $l - \epsilon < y_n < l + \epsilon$ for all $\geq k$, which gives , $|y_n - l| < \epsilon$ for all $n \geq k$.

This shows that the sequence $\{y_n\}$ convergent and converges to . ■

Note : The above theorem is also true for the sequence of real numbers $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ such that $x_n \leq y_n \leq z_n$ for all $n \in N$.

Null sequence : A sequence $\{x_n\}$ is said to be a null sequence if $\lim x_n = 0$.

Theorem 3.6: If $\{x_n\}$ is a null sequence then $\{|x_n|\}$ is also a null sequence and conversely .

Proof : Since $\{x_n\}$ is a convergent sequence and converging to zero , by the **Absolutely convergence theorem** $\{|x_n|\}$ is also convergent and converges to $|0| = 0$.

Conversely let , $\lim |x_n| = 0$. Let us choose a positive number $\epsilon > 0$. There exists a natural number k such that ,

$$|x_n - 0| = |x_n| = |\,|x_n| - 0\,| < \epsilon \quad \text{,for all } n \geq k .$$

This proves $\lim x_n = 0$. ■

Example 3.2: $\lim_{n\to\infty} \frac{n^2+3n}{2n^2+n-1} = \frac{1}{2}$.

Solution : $\lim_{n\to\infty} \frac{n^2+3n}{2n^2+n-1} = \lim_{n\to\infty} \frac{1+3/n}{2+1/n-1/n^2} = \frac{\lim(1+3/n)}{\lim(2+1/n-1/n^2)}$

$$= \frac{1+ \lim 3/n}{2+ \lim 1/n - \lim 1/n^2} = \frac{1+0}{2+0-0} = \frac{1}{2} .$$

Example 3.3: $\lim\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)=1$.

Solution : Let $x_n = \frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}$.

We have , $\frac{1}{\sqrt{n^2+2}}<\frac{1}{\sqrt{n^2+1}}$, $\frac{1}{\sqrt{n^2+2}}<\frac{1}{\sqrt{n^2+1}}$,….. , , $\frac{1}{\sqrt{n^2+n}}<\frac{1}{\sqrt{n^2+1}}$

Therefore , $x_n < \frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+1}}+\cdots+\frac{1}{\sqrt{n^2+1}} = \frac{n}{\sqrt{n^2+1}}$,for all $n \geq 2$.

Again , $\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}} > \frac{2}{\sqrt{n^2+2}}$

$\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\frac{1}{\sqrt{n^2+3}} > \frac{3}{\sqrt{n^2+3}}$ and so on .

Therefore , $x_n > \frac{n}{\sqrt{n^2+n}}$ for all $n \geq 2$.

Thus , $\frac{n}{\sqrt{n^2+1}} < x_n < \frac{n}{\sqrt{n^2+n}}$ for all $n \geq 2$.

Now , $\lim \frac{n}{\sqrt{n^2+n}} = \lim \frac{1}{\sqrt{1+1/n}} = 1$ and $\lim \frac{n}{\sqrt{n^2+1}} = \lim \frac{1}{\sqrt{1+1/n^2}} = 1$.

Therefore by the Sandwich theorem , $\lim x_n = 1$.

Section 4 : Divergent Sequence .

(i) A real sequence $\{x_n\}$ is said to **diverge to** $+\infty$,if for every $M > 0$, however large , there exists a natural number k such that ,

$$x_n > M \quad \text{for all } n \geq k .$$

In this case we write , $\lim x_n = \infty$.

(ii) A real sequence $\{x_n\}$ is said to **diverge to** $-\infty$,if for every $M > 0$, however large , there exists a natural number k such that ,

$$x_n < -M \quad \text{for all } n \geq k .$$

In this case we write , $\lim x_n = -\infty$.

A real sequence $\{x_n\}$ is said to be **properly diverges** if it either diverges to $+\infty$ or $-\infty$.

Theorem 4.1:
(a) A sequence diverging to $+\infty$ is unbounded above but bounded below .

(b) A sequence diverging to $-\infty$ is unbounded below but bounded above .

Proof : Proof is left as an exercise . ■

Note : Converse of the above theorem is not true i.e a sequence unbounded above but bounded below may not diverges to $+\infty$ and a sequence unbounded below but bounded above may not diverges to $-\infty$.

For example ,consider the sequence $\{n^{(-1)^n}\}$ i.e. $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, \ldots\}$, The sequence is unbounded above and bounded below , 0 is a lower bound . But the sequence does not diverges to $+\infty$.

<u>Alternating sequence</u> : If a sequence $\{x_n\}$ is such that , $x_n . x_{n+1} < 0$ for all $n \in N$, that is $\{x_n\}$ changes sign alternately , then we say $\{x_n\}$ is a alternating sequence .

An alternating sequence converge or diverge . For example the sequence $\{(-1)^n\}$ diverges , whereas $\{(-1)^n/n\}$ converges to 0 .

<u>Oscillatory sequence</u> : A sequence which neither converges nor diverges to $+\infty$ or $-\infty$ is called oscillatory sequence .

A bounded sequence that is not convergent is called an oscillatory sequence of **finite oscillation** . An unbounded sequence that is not properly divergent is called an oscillatory sequence of **infinite oscillation** .

Examples 4.1: (1) The sequences $\{2n\}$,$\{3^n\}$, $\{x^n\}$ where $x > 1$ diverges to $+\infty$.

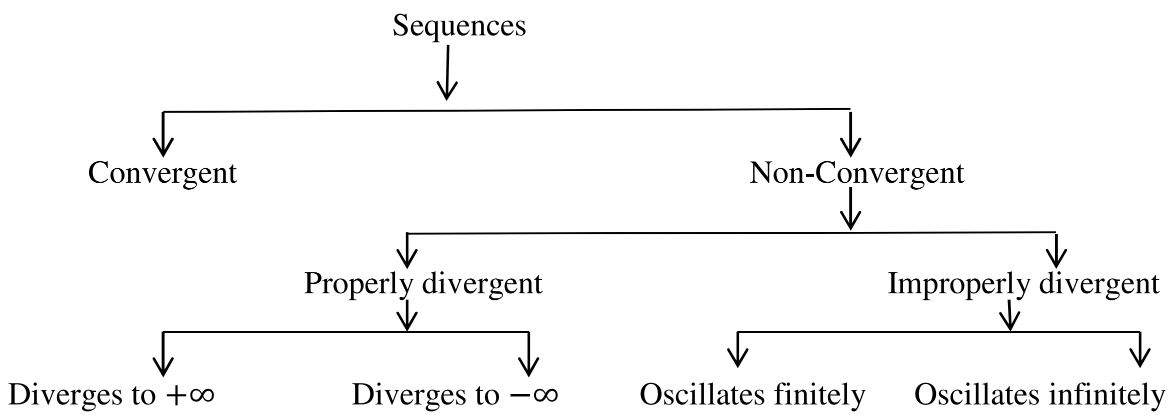
(2) The sequences $\{-2n\}$,$\{-n^2\}$ diverges to $-\infty$.

(3) The sequence $\{(-1)^n\}$ is not convergent and also does not diverges properly .It is an oscillatory sequence with finite oscillation . It oscillate between -1 and 1 .

(4) The sequence $\{(-1)^n\, n\}$ is an unbounded sequence and it is not properly divergent .It is an oscillatory sequence of infinite sequence .

(5) The sequence $\{1 + (-1)^n\}$ oscillates finitely between 0 and 2 .

■ The following model can be easily remembered about the behavior of a sequence with respect to convergence or divergence :



■ Some useful limits :

Example 4.2: Let $|a| < 1$, then $\lim a^n = 0$.

Solution : Let $a \neq 0$, because $\lim a^n = 0$ is clear for $a = 0$.

Now $\frac{1}{|a|} > 1$, since $|a| < 1$. We may write $\frac{1}{|a|} = 1 + b$ where $b > 0$ i.e. $|a| = \frac{1}{1+b}$.

By the Bionomial theorem we have ,

$$(1 + b)^n = 1 + nb + \frac{n(n-1)}{2} b^2 + \cdots + b^n > nb \text{ for all } n \in N .$$

So ,

$$|a^n - 0| = |a^n| = \frac{1}{(1+b)^n} < \frac{1}{nb} \text{ for all } n \in N .$$

Therefore , $|a^n - 0| < \epsilon$ holds if $\frac{1}{nb} < \epsilon$ i.e. if $n > 1/b\epsilon$.

Let $= \left[\frac{1}{b\epsilon}\right] + 1$. Then *k* is a natural number such that , $|a^n - 0| < \epsilon$ for all $n \geq k$.

Hence, $\lim a^n = 0$.

Example 4.3 : $\lim n^{1/n} = 1$.

Solution : Let $a_n = n^{1/n} - 1$ and note that $a_n > 0$ for all $n \geq 2$. To prove the limit it is sufficient to show that $\lim a_n = 0$.

Since $1 + a_n = n^{1/n}$,we have $n = (1 + a_n)^n$. For $n \geq 2$, by the bionomial theorem we have ,

$$n = (1 + a_n)^n \geq 1 + na_n + \frac{1}{2} n(n-1)a_n^2 > \frac{1}{2} n(n-1)a_n^2 .$$

Thus , $n > \frac{1}{2}n(n-1)a_n^2$, so $a_n^2 < \frac{2}{n-1}$. Thus , we have shown that $0 < a_n < \sqrt{\frac{2}{n-1}}$ for all $n \geq 2$.

Since $\lim \sqrt{\frac{2}{n-1}} = 0$, by Sandwich theorem we get $\lim a_n = 0$. Thus , $\lim n^{1/n} = 1$.

Example 4.4 : $\lim a^{1/n} = 1$ if $a > 0$.

Solution : Suppose $a \geq 1$. Then for $n \geq a$,we have $1 \leq a^{1/n} \leq n^{1/n}$. Since $\lim n^{1/n} = 1$,it is easily follows that $\lim a^{1/n} = 1$. Suppose that $0 < a < 1$. Then $1/a > 1$, so that $\lim \left(\frac{1}{a}\right)^{1/n} = 1$. Thus ,

$$\lim \left(\frac{1}{a}\right)^{1/n} = \lim \frac{1^{1/n}}{a^{1/n}} = \frac{\lim 1^{1/n}}{\lim a^{1/n}} = \frac{1}{\lim a^{1/n}} = 1.$$

This gives , $\lim a^{1/n} = 1$.

Theorem 4.2 : Let $\{a_n\}$ and $\{b_n\}$ are two sequences such that $\lim a_n = +\infty$ and $\lim b_n > 0$. Then $\lim a_n b_n = +\infty$.

Proof : Let $M > 0$. Choose a real number m so that , $0 < m < \lim b_n$. Whether $\lim b_n = +\infty$ or not ,there exists a natural number k_1 such that $b_n > m$ for all $n \geq k_1$. Since $\lim a_n = +\infty$, there exists a natural number k_2 such that ,

$$a_n > M/m \text{ ,for all } n \geq k_2 .$$

Let $= \max\{k_1 , k_2\}$. Then we have ,

$$a_n > M/m \text{ and } b_n > m \text{ ,for all } n \geq k \quad .$$

Which gives , $a_n b_n > \frac{M}{m} . m = M$, for all $n \geq k$. Therefore , $\lim a_n b_n = +\infty$. ■

Theorem 4.3: For a sequence $\{a_n\}$ of positive real numbers $\lim a_n = +\infty$ if and only if $\lim 1/a_n = 0$.

Proof : Let $\{a_n\}$ be a sequence of positive real numbers . Suppose that , $\lim a_n = +\infty$.

Let $\epsilon > 0$ and $= 1/\epsilon$. Since $\{a_n\}$ diverges to $+\infty$,there exists a natural number k_1 such that , $a_n > M = 1/\epsilon$ for all $n \geq k_1$. Therefore , if $n \geq k$ then $0 < 1/a_n < \epsilon$, so

$$\left|\frac{1}{a_n} - 0\right| = 1/a_n < \epsilon \quad \text{for all } n \geq k_1 .$$

Thus , $\lim 1/a_n = 0$.

Conversely let , $\lim 1/a_n = 0$ and $M > 0$. Let $\epsilon = 1/M$ .Then since $\epsilon > 0$ there exists a natural number k_2 such that ,

$$\left|\frac{1}{a_n} - 0\right| < \epsilon = 1/M \quad \text{for all } n \geq k_2 .$$

Since $a_n > 0$, we have $1/a_n < 1/M$,for all $n \geq k_2$. Which gives , there exists a natural number k_2 such that , $a_n > M$ for all $n \geq k_2$.

Thus , $\lim a_n = +\infty$. ■

We will now complete this section by stating two useful theorem without proof . Proof can be found in any standard book.

Theorem 4.4: (Ratio test) Let $\{a_n\}$ be a sequence of positive real numbers such that , $\lim \frac{a_{n+1}}{a_n} = l$. If $0 \leq l < 1$ then $\lim a_n = 0$ and if $l > 1$ then $\lim a_n = \infty$.

Theorem 4.5: (Root test) Let $\{a_n\}$ be a sequence of positive real numbers such that , $\lim (a_n)^{1/n} = l$ .If $0 \leq l < 1$ then $a_n = 0$ and if $l > 1$ then $\lim u_n = \infty$. ■

Until now, we have obtained several methods of showing that a sequence $\{x_n\}$ of real numbers is convergent . all of these methods require that we already know (or at least suspect) the value of the limit, and we then verify that our suspicion is correct.

There are many instances, however, in which there is no obvious candidate for the limit of a sequence, even though a preliminary analysis may suggest that convergence is likely. In this and the next two sections, we shall establish results that can be used to show a sequence is convergent even though the value of the limit is not known. The method we introduce in this section is more restricted in scope than the methods we give in the next two, but it is much easier to employ. It applies to sequences that are monotone in the following sense.

Definition : Consider a sequence real number $\{x_n\}$.

(i) $\{x_n\}$ is said to be a **monotone increasing sequence** if $x_{n+1} \geq x_n$ for all $n \in N$.

(ii) $\{x_n\}$ is said to be a **monotone decreasing sequence** if $x_{n+1} \leq x_n$ for all $n \in N$.

If strict inequality occurs in (i) and (ii) ,then we say we say that sequence is **strictly increasing** and **strictly decreasing** respectively .

A sequence is $\{x_n\}$ is said to be simply **monotone** ,if it is either monotone increasing or monotone decreasing .

Examples 5.1: **(1)** Consider the sequence $\{x_n\}$ where $x_n = 2^n$ i.e. $\{2, 4, 8 , \ldots\ldots\}$. Then , $x_{n+1} > x_n$ for all $n \in N$. Therefore the sequence $\{x_n\}$ is a monotone increasing sequence . It is also strictly monotone increasing .

(2) Consider the sequence $\{x_n\}$ where , $x_n = 1/n$, i.e. $\{1 , \frac{1}{2}, \frac{1}{3}, \ldots\ldots\}$. Then , $x_{n+1} < x_n$ for all $n \in N$. Therefore the sequence $\{x_n\}$ is a monotone decreasing sequence . It is also strictly monotone decreasing .

(3) The sequence $\{(-2)^n\}$ i.e. $\{-2 , 2 , -2 , 2 , \ldots\}$ is neither monotone increasing sequence nor monotone decreasing sequence .

Theorem 5.1: (i) A monotone increasing sequence ,if bounded above ,is convergent and it converges to the least upper bound .

(ii) A monotone decreasing sequence , if bounded below , is convergent and it converges to the greatest lower bound .

Proof : (i) Let $\{x_n\}$ be a monotone increasing sequence which is bounded above and *M* is the least upper bound of $\{x_n\}$. Then ,
(a) $x_n \leq M$ for all $n \in N$ and

(b) for a pre-assigned $\epsilon > 0$,there exists a natural number k such that , $x_k > M - \epsilon$.

Since $\{x_n\}$ is a monotone increasing sequence ,

$$M - \epsilon < x_k < x_{k+1} < x_{k+2} < \cdots \leq M < M + \epsilon$$

That is , $M - \epsilon < x_n < M + \epsilon$ for all $n \geq k$. This shows that the sequence $\{x_n\}$ is convergent and converges to M .

(ii) Left as an exercise . ■

Monotone Convergence Theorem : A monotone sequence of real numbers is convergent if and only if it is bounded .

Proof : By using the previous Theorem 5.1 one can easily prove the result . It is left as an exercise. ■

Theorem 5.2: (i) A monotone increasing sequence that is unbounded above diverges to $+\infty$.

(ii) A monotone decreasing sequence that is unbounded below diverges to $-\infty$.

Proof : (i) Let $\{x_n\}$ be a monotone increasing sequence which is unbounded above . Since the is unbounded above , for every $M > 0$,however large , there exists a natural number k such that $x_k > M$.

Since $\{x_n\}$ is monotone increasing sequence , we have

$$M < x_k < x_{k+1} < x_{k+2} < \cdots$$

That is , $x_n > M$ for all $n \geq k$. This proves that $\{x_n\}$ diverges to $+\infty$.

(ii) Left as an exercise . ■

Remark : A monotone sequence has a definite behavior .It is either convergent or properly divergent . That is a monotone sequence can never oscillate .

Example 5.2: The sequence $\{1/n\}$ is monotone decreasing sequence and bounded below , zero being a lower bound . Therefore $\{1/n\}$ is convergent and converges to the greatest lower bound ,which is 0 .

Example 5.3: If $x_n = \frac{3n-1}{n+2}$, prove that $\{x_n\}$ is monotone increasing and bounded .

Solution : $x_{n+1} - x_n = \frac{3(n+1)-1}{(n+1)+2} - \frac{3n-1}{n+2} = \frac{3n+2}{n+3} - \frac{3n-1}{n+2} = \frac{7}{(n+3)(n+2)} > 0$,for all n.

So , $x_{n+1} > x_n$ for all $n \in N$. Therefore $\{x_n\}$ is a monotone increasing sequence .

Also , $x_n = \frac{3n-1}{n+2} = \frac{3(n+2)-7}{n+2} = 3 - \frac{7}{n+2} < 3$ for all $n \in N$.

Thus the sequence is bounded above . Also being a monotone increasing sequence $\{x_n\}$ certainly a bounded below sequence (because , $x_n > x_1 = 2/3$) . Hence $\{x_n\}$ is bounded .

Since $\{x_n\}$ is monotone increasing and bounded above , by the previous theorem we get $\{x_n\}$ is convergent and converges to least upper bound . ■

Section 6 : Subsequence .

In this section we will introduce the notion of a subsequence of a sequence of real numbers. Informally, a subsequence of a sequence is a selection of terms from the given sequence such that the selected terms form a new sequence. Usually the selection is made for a definite purpose. For example, subsequences are often useful in establishing the convergence or the divergence of the sequence. We will also prove the important existence theorem known as the Bolzano-Weierstrass Theorem, which will be used to establish a number of significant results.

Definition : Let $\{x_n\}$ be a sequence of real numbers and $\{r_n\}$ be a strictly increasing sequence of natural numbers i.e. $r_1 < r_2 < r_3 < \cdots < r_n < \cdots$. Then the sequence $\{x_{r_n}\}$ is said to be a **subsequence** of the sequence $\{x_n\}$. The elements of the subsequence $\{x_{r_n}\}$ are x_{r_1}, x_{r_2}, x_{r_3} , … , x_{r_n} , … . .

With the concept of subset one can relate the concept of the subsequence . Like subset the elements of the subsequence $\{x_{r_n}\}$ are nothing but the elements of $\{x_n\}$ which are chosen in some proper way. So the range set $\{x_{r_n}: n \in N\}$ of the subsequence is the subset of the range set $\{x_n : n \in N\}$ of the sequence $\{x_n\}$.

Example 6.1 :(1) Let $x_n = 1/n$ and $r_n = 2n$. Then $\{x_{r_n}\} = \{x_2, x_4, x_6, \ldots\} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\}$ is a sequence of $\{x_n\}$.

(2) Let $x_n = (-1)^n$. Then $\{x_{2n}\} = \{x_2, x_4, x_6, \ldots.\} = \{1, 1, 1, 1, \ldots.\}$ and $\{x_{2n-1}\} = \{x_1, x_3, x_5, \ldots.\} = \{-1, -1, -1, \ldots.\}$ are two subsequences of $\{x_n\}$.

A sequence may not converge, but it can have convergent subsequences. For example, we know that the sequence $\{(-1)^n\}$ diverges, but the subsequences $\{1, 1, 1, \ldots\}$ and $\{-1, -1, -1, \ldots\}$ are convergent subsequence of $\{(-1)^n\}$. However we have the following result :

Theorem 6.1: If a sequence $\{x_n\}$ converges to *x* ,then every subsequence of $\{x_n\}$ is also converges to *x* .

Proof : Let $\{x_{r_n}\}$ be a subsequence of $\{x_n\}$. Also let us chose a > 0 . Since $\{x_n\}$ converges to *x*, therefore there exists a natural number *k* such that, $|x_n - x| < \epsilon$ for all $n \geq k$.

Since $\{r_n\}$ is a strictly increasing sequence of natural number , there exists a natural number k_1 such that , $r_n > k$ for all $n \geq k_1$ (since k depends on ϵ , k_1 is also depends on ϵ)

Therefore ,$|x_{r_n} - x| < \epsilon$ for all $n \geq k_1$. This shows that , $\lim x_{r_n} = x$. Hence proved . ■

Remark : If a sequence $\{x_n\}$ of real numbers has either of the following properties ,then $\{x_n\}$ is divergent –

(i) If $\{x_n\}$ has two subsequences $\{x_{r_n}\}$ and $\{x_{k_n}\}$ whose limits are not equal .

(ii)If $\{x_n\}$ has a divergent subsequence .

(iii) If $\{x_n\}$ is unbounded .

Example 6.2 : The sequence $\{(-1)^n\}$ i.e. $\{-1, 1 , -1 ,1 , ...\}$ is divergent .

The subsequence $\{(-1)^{2n}\}$ i.e. $\{1, 1 , 1 , ...\}$ converges to 1 and the subsequence $\{(-1)^{2n-1}\}$ i.e. $\{-1 , -1 , -1 , ...\}$ converges to -1 . Since two different subsequences converges to different limit , so by the above result we may conclude that $\{(-1)^n\}$ diverges .

■ What about the converse of the above **Theorem 6.1**? Obviously, if all subsequences of a sequence $\{x_n\}$ converge to the same limit *x*, then $\{x_n\}$ also has to converge to *x*, as $\{x_n\}$ is a subsequence of itself.

Suppose every subsequence of $\{x_n\}$ has at least one subsequence which converges to *x*. Does the sequence $\{x_n\}$ converges to *x*? The answer is affirmative, as the following theorem shows.

Theorem 6.3: If every subsequence of $\{x_n\}$ has at least one subsequence which converges to *x*, then $\{x_n\}$ also converges to *x*.

Proof : Left as an exercise . ■

Theorem 6.4: If the subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ of a sequence $\{x_n\}$ converge to the same limit *x* ,then the sequence $\{x_n\}$ is also convergent and converges to *x* .

Note : If two subsequences of a sequence converge to the same limit ,the sequence $\{x_n\}$ may not be convergent . For example consider the sequence $\{x_n\} = \{\sin\frac{n\pi}{4}\}$. The subsequence $\{x_{8n-7}\}$ and $\{x_{8n-5}\}$ converges to $1/\sqrt{2}$, but the sequence $\{x_n\}$ is not convergent .

Theorem 6.5: Every subsequence of a monotone increasing (decreasing) sequence of real numbers is monotone increasing (decreasing) .

Proof : Left as an exercise . ■

Theorem 6.4 : A monotone sequence of real numbers having a convergent subsequence with limit l , is convergent with limit l .

Proof : Let $\{x_n\}$ be a monotone increasing sequence and $\{x_{r_n}\}$ is a subsequence of $\{x_n\}$ such that , $\lim x_{r_n} = l$. Then $\{x_{r_n}\}$ is monotone increasing and being a convergent sequence ,it is bounded above also.

Now we assert that $\{x_n\}$ is bounded above . If possible let ,$\{x_n\}$ is unbounded above . Then for a real number $M > 0$,however large ,there exists a natural number $k \in N$ such that , $x_n > M$ for all $n \geq k$.

Since $\{r_n\}$ is a strictly increasing sequence of natural number , therefore there exists a natural number k_1 such that , $r_n > k$ for all $n \geq k_1$. Consequently , $x_{r_n} > M$ for all $n \geq k_1$.

Since M is arbitrary , we have $\lim x_{r_n} = +\infty$. Which is a contradiction .So our assumption is wrong i.e. $\{x_n\}$ is bounded above . Also since $\{x_n\}$ is monotone , it is convergent .

Now let , $\lim x_n = m$. Then $\{x_{r_n}\}$ being a subsequence of $\{x_n\}$,it is also convergent to m . Therefore $= m$. Thus $\{x_n\}$ is also converges to . ■

The Existence of Monotone Subsequence :

While not every sequence is a monotone sequence, we will now show that every sequence has a monotone subsequence.

Monotone Subsequence Theorem : Every sequence of real numbers has a monotone subsequence .

In order to prove this theorem let us introduce what we call a peak of a sequence:

Peak of a sequence :

Let $\{x_n\}$ is a sequence of real numbers . An element x_k is said to be a peak of the sequence $\{x_n\}$ if , $x_k \geq x_n$ for all $n > k$ i.e. x_k is greater or equal to all subsequent elements beyond x_k . A sequence may or may not have a peak or else it may have a finite or infinite number of peaks .

Example 6.3: **(1)** Consider the sequence $\{x_n\} = \{(-1)^n\}$ i.e. $\{-1, 1, -1, 1, ...\}$ .Here $x_2 = 1 \geq x_n$ for all > 2 . Therefore x_2 is a peak . Similarly x_4 , x_6 are also peaks . Actually the sequence $\{x_n\}$ has infinitely many peaks which are $x_2, x_4, x_6, x_8, ...$.

Also the collection of the peaks $\{x_2, x_4, x_6, x_8, ...\}$ is monotone subsequence of $\{x_n\}$.

(2) Let $x_n = n^{(-1)^n}$ .The sequence is $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, ...\}$. Here the sequence has no peak .

Proof of Monotone subsequnce theorem : We consider the following cases .

Case – I : Let $\{x_n\}$ have infinitely many peaks . Let the peaks are $x_{r_1}, x_{r_2}, x_{r_3}, \ldots$ (where $\{r_1, r_2, r_3, \ldots\ldots\}$ is increasing sequence of natural number .) i.e. x_{r_1} is the first peak ,x_{r_2} is the second peak and so on . Then , $x_{r_1} \geq x_{r_2} \geq x_{r_3} \geq \cdots$. The subsequence $\{x_{r_n}\}$ is a monotone decreasing subsequence .

Case – II : Let $\{x_n\}$ has finite number (possibly zero) of peaks . Let these peaks be listed by increasing subscripts : $x_{r_1}, x_{r_2}, \ldots\ldots, x_{r_m}$. Let $s_1 = r_m + 1$ be the first index beyond the peak . Then x_{s_1} is not a peak and no peak beyond x_{s_1} .

Since x_{s_1} is not a peak ,there exists a natural number $s_2 > s_1$ such that $x_{s_2} > x_{s_1}$. Again since x_{s_2} is not a peak ,there exists a natural number $s_3 > s_2$ such that , $x_{s_3} > x_{s_2}$. Continuing this way we obtained a strictly increasing sequence of natural number $\{s_n\}$ and a monotone increasing subsequence $\{x_{s_n}\}$ of $\{x_n\}$. ■

Note : It is not difficult to see that a given sequence may have one subsequence that is increasing, and another subsequence that is decreasing. As example ,the sequence $\{n^{(-1)^n}\}$ i.e $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \ldots.\}$ has two subsequences, $\{2, 4, 6, \ldots\}$,which is monotone increasing and $\{1, \frac{1}{3}, \frac{1}{5}, \ldots\}$, which is monotone decreasing .

<u>The Bolzano-Weierstrass Theorem</u>

We will now use the Monotone Subsequence Theorem to prove the Bolzano-Weierstrass Theorem, which states that every bounded sequence has a convergent subsequence. This theorem can also be proved by using Nested Interval Property , One can find this proof in any standard text book .

The Bolzano-Weiertrass Theorem : Every bounded sequence of real numbers has a convergent subsequence .

Proof : Let $\{x_n\}$ be a bounded sequence of real numbers . It follows from the Monotone Subsequence Theorem that , $\{x_n\}$ has a monotone subsequence $\{x_{r_n}\}$. Since $\{x_n\}$ is bounded , then the subsequence $\{x_{r_n}\}$ of $\{x_n\}$ is also bounded .

Since $\{x_{r_n}\}$ is a monotone and bounded subsequence ,then from Monotone Convergence Theorem we have $\{x_{r_n}\}$ is convergent . Therefore $\{x_{r_n}\}$ is convergent subsequence of $\{x_n\}$. This proves our result . ■

Note : A unbounded sequence may have a convergent subsequence . For example , consider the sequence $\{n^{(-1)^n}\}$ i.e. $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \ldots\}$. It is unbounded above but it has a convergent subsequence $\{1, \frac{1}{3}, \frac{1}{5}, \ldots\}$,converges to 0 .

Section 7: Limit Superior and Limit Inferior .

A bounded sequence of real numbers $\{x_n\}$ may or may not converge, but we know from the Bolzano-Weierstrass Theorem that there will be a convergent subsequence and possibly many convergent subsequences. We now introduce the concept of subsequential limit .

Subsequential Limit : Let $\{x_n\}$ is a real sequence . A real number x is said to be a subsequential limit of the sequence $\{x_n\}$ if there exists a subsequence of $\{x_n\}$ converges to x . Note that the limit of a sequence ,if it exists ,is also a subsequential limit of the sequence .

Example 7.1: (1) Consider the sequence $\{(-1)^n\}$. Then the subsequences $\{1,1,1 \dots\}$ and $\{-1,-1,-1,\dots\}$ converges to 1 and -1 respectively . Therefore 1 and -1 are subsequential limits of $\{x_n\}$.

(2) Consider the sequence $\{n^{(-1)^n}\}$ i.e. $\{1,2,\frac{1}{3},4,\frac{1}{5},6,\dots\}$ Then the subsequences $\left\{1,\frac{1}{3},\frac{1}{5}\dots\right\}$ converges to 0 . Therefore 0 is a subsequential limit of $\{n^{(-1)^n}\}$.

Theorem 7.1: A real number x is a subsequential limit of $\{x_n\}$ if and only if every neighbourhood of x contains infinitely many elements of $\{x_n\}$.

Proof : Left as an exercise .

■ Let $\{x_n\}$ be a bounded sequence of real numbers .Then by Bolzano-Weiertrass theorem there is a subsequential limit of $\{x_n\}$. Let S denote the set of all subsequential limits of $\{x_n\}$. Since $\{x_n\}$ is bounded , the set S is also bounded .

If S is finite , then S has a greatest element and a least element . Also if S is an infinite set , then being a bounded set it can be proved that S has a greatest as well as least element .

Definition : Let $\{x_n\}$ be a bounded sequence of real numbers .

(a) The greatest subsequential limit of $\{x_n\}$ is said to be **upper limit or limit superior** of $\{x_n\}$ and it is denoted by $\overline{\lim}\, x_n$ or lim sup x_n .

(b) The least subsequential limit of $\{x_n\}$ is said to be **lower limit or limit inferior** of $\{x_n\}$ and it is denoted by $\underline{\lim}\, x_n$ or lim inf x_n .

Note : If $\{x_n\}$ is unbounded above then we define , $\overline{\lim}\, x_n = \infty$ and if unbounded below then we define , $\underline{\lim}\, x_n = -\infty$.

Example 7.2: (1) Let $x_n = (-1)^n$. Then the sequence $\{x_n\}$ is bounded and the set of subsequential limits $= \{1,-1\}$. Therefore $\overline{\lim}\, x_n = 1$ and $\underline{\lim}\, x_n = -1$.

(2) Let $x_n = 1/n$. Then the sequence $\{x_n\}$ is bounded and 0 is the only subsequential limit i.e. $S = \{0\}$. Therefore $\overline{\lim}\, x_n = \underline{\lim}\, x_n = 0$.

(3) Let $x_n = n^{(-1)^n}$. Then $\{x_n\}$ is unbounded above and bounded below and $S = \{0\}$. Therefore $\overline{\lim}\, x_n = \infty$ and $\underline{\lim}\, x_n = 0$.

(4) Let $x_n = (-1)^n n^2$. Then $\{x_n\}$ is unbounded above and unbounded below and Therefore $\overline{\lim}\, x_n = \infty$ and $\underline{\lim}\, x_n = -\infty$.

■ We will now state two useful theorem about **limit superior** and **limit inferior** without proof.

Theorem 7.2: If $\{x_n\}$ is a bounded sequence of real numbers, then the following statements for a real number x^* are equivalent.

(a) $x^* = \lim\sup\, x_n$.

(b) For > 0 , $x_n > x^* - \epsilon$ for infinitely many values of *n* and there exists a natural number *k* such that $x_n < x^* + \epsilon$ for all $n \geq k$.

(c) If *S* is the set of subsequential limits of $\{x_n\}$,then $x^* = \sup S$.

Theorem 7.3: If $\{x_n\}$ is a bounded sequence of real numbers, then the following statements for a real number x_* are equivalent.

(a) $x_* = \lim\inf\, x_n$.

(b) For > 0 , $x_n < x_* + \epsilon$ for infinitely many values of *n* and there exists a natural number *k* such that $x_n > x_* - \epsilon$ for all $n \geq k$.

(c) If *S* is the set of subsequential limits of $\{x_n\}$,then $x_* = \inf S$.

■ By using these above two theorems we will now prove the following useful result .

Theorem 7.4: A bounded sequence $\{x_n\}$ is convergent if and only if $\overline{\lim}\ x_n = \underline{\lim}\ x_n$.

Proof : Let $\{x_n\}$ is a convergent sequence and $\lim x_n = x$. Since $\{x_n\}$ is converges to x , every subsequence of $\{x_n\}$ converges to x . Therefore x is the greatest as well as least subsequential limit . That is , $\overline{\lim}\ x_n = \underline{\lim}\ x_n$.

Conversely let , $\{x_n\}$ be a bounded sequence such that $\overline{\lim}\ x_n = \underline{\lim}\ x_n = x$.

Since $\overline{\lim}\ x_n = x$, for $\epsilon > 0$ there exists a natural number k_1 such that , $x_n < x + \epsilon$ for all $n \geq k_1$.

Again since $\underline{\lim}\ x_n = x$, for $\epsilon > 0$ there exists a natural number k_2 such that , $x_n > x - \epsilon$ for all $n \geq k_2$.

Let $= \max\{k_1 , k_2\}$. Then , $x - \epsilon < x_n < x + \epsilon$ for all $n \geq k$. This proves that , $\lim x_n = x$. That is $\{x_n\}$ converges . ■

The above theorem can be restated as – 'A bounded sequence is convergent if and only if it has only one subsequential limit' . Next we will state a useful theorem without proof .

Theorem 7.5 : Let $\{x_n\}$ and $\{y_n\}$ are bounded sequences . Then ,

(i) $\overline{\lim}\ x_n + \overline{\lim}\ y_n \geq \overline{\lim}\ (x_n + y_n)$

(ii) $\underline{\lim}\ x_n + \underline{\lim}\ y_n \leq \underline{\lim}\ (x_n + y_n)$.

Section 8 : The Cauchy Criterion: .

So far we have discussed several methods of establishing convergence of a real sequence .In most of the methods , a prior knowledge of the limit is necessary . If however a sequence is monotone , the convergence can be established without any pre-conceived limit .

The Monotone Convergence Theorem is extraordinarily useful and important, but it has the significant drawback that it applies only to sequences that are monotone. It is important for us to have a condition implying the convergence of a sequence that does not require us to know the value of the limit in advance, and is not restricted to monotone sequences. The Cauchy Criterion, which will be established in this section, is such a condition

Definition : A sequence $\{x_n\}$ is said to be a **Cauchy sequence** if every $\epsilon > 0$ there exists a natural number k (depends on ϵ) such that ,

$$|x_m - x_n| < \epsilon \ \text{ for all } m , n \geq k .$$

Replacing m by $n + p$ where $p = 1, 2 , 3 , ...$ the above condition can be equivalently stated as ,

$$|x_{n+p} - x_n| < \epsilon \ \text{ for all } n \geq k .$$

Example 8.1 :

(1) The sequence $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence .

Let $x_n = 1/n$. For a pre-assigned > 0 , we choose a natural number k , large enough , such that $2/k < \epsilon$. Then ,

$$|x_m - x_n| = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{m} + \frac{1}{n} \leq \frac{1}{k} + \frac{1}{k} = \frac{2}{k} < \epsilon \quad ,\text{if } m , n \geq k .$$

This proves that the sequence $\{x_n\}$ is Cauchy sequence .

(2) The sequence $\{1 + (-1)^n\}$ is not Cauchy .

Let $x_n = 1 + (-1)^n$. Then $|x_m - x_n| = |(-1)^m - (-1)^n|$. Therefore ,

$$|x_m - x_n| = 0 \quad \text{, if } m \text{ and } n \text{ are both odd or even .}$$
$$= 2 \quad \text{, if one of m , n is odd and the other is even .}$$

Let us choose $= 1 > 0$. Then it is not possible to find a natural number k such that , $|x_m - x_n| < \epsilon$. This proves the sequence $\{x_n\}$ is not Cauchy .

■ Our goal is to show that the Cauchy sequences are precisely the convergent sequences. We first prove that a convergent sequence is a Cauchy sequence.

Theorem 8.1: A convergent sequence is a Cauchy sequence .

Proof : Let $\{x_n\}$ be a convergent sequence and $\lim x_n = x$. Then for given $\epsilon > 0$,there exists a natural number k such that , $|x_n - x| < \epsilon/2$ for all $n \geq k$.

If $n, m \geq k$ then , $|x_n - x| < \epsilon/2$ and $|x_m - x| < \epsilon/2$. Therefore ,

$$|x_m - x_n| = |x_m - x + x - x_n| \leq |x_m - x| + |x_n - x|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon \text{ for all } n, m \geq k .$$

This proves that $\{x_n\}$ is a Cauchy sequence . ■

Theorem 8.2: A Cauchy sequence of real numbers is bounded .

Proof : Let $\{x_n\}$ be a Cauchy sequence and let $\epsilon = 1$. Then there exists a natural number k such that , $|x_m - x_n| < \epsilon = 1$ for all $n, m \geq k$. Taking $m = k$,we have $|x_n - x_k| < 1$ for all $n \geq k$.

Therefore by Triangle inequality , $|x_n| = |x_n - x_k + x_k| \leq |x_n - x_k| + |x_k| < 1 + |x_k|$, for all $n \geq k$. If we set ,

$$M = \max\{ |x_1| , |x_2| , \dots , |x_{k-1}| , 1 + |x_k| \}$$

This proves $\{x_n\}$ is a bounded sequence . ■

Cauchy Convergence Criterion : A sequence of real numbers is convergent if and only if it is a Cauchy sequence .

Proof : We have seen , in Theorem 8.1 , that a convergent sequence is Cauchy .

Conversely let $\{x_n\}$ is a Cauchy sequence . We will show that $\{x_n\}$ is convergent to some real number.

Since $\{x_n\}$ is Cauchy , then by the previous theorem $\{x_n\}$ is bounded . Therefore by Bolzano-Weiertrass theorem $\{x_n\}$ has a converges subsequence and let the subsequence converges to x .

Let us choose > 0 . Since $\{x_n\}$ is Cauchy ,there exists a natural number k such that ,

$$|x_m - x_n| < \epsilon/2 \quad \text{for all } n, m \geq k .$$

Since x is a subsequential limit , then by theorem 7.1 the neighborhood $N_{\frac{\epsilon}{2}}(x)$ of x contains infinitely many elements of $\{x_n\}$. Therefore there exists a natural number q such that , $|x_q - x| < \epsilon/2$.

Since $q > k$, we have $|x_q - x_n| < \epsilon/2$ for all $n \geq k$. Therefore ,

$$|x_n - x| = |x_n - x_q + x_q - x| \leq |x_n - x_q| + |x_q - x|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon \quad \text{, for all} \geq k .$$

This implies that the sequence $\{x_n\}$ converges to x . Hence proved . ■

Definition : A sequence is $\{x_n\}$ of real numbers is said to be **contractive** if there exists a constant $0 < C < 1$ such that ,

$$|x_{n+2} - x_{n+1}| \leq C\,|x_{n+1} - x_n| \quad \text{for all } n \in N .$$

Exercise 8.1: Prove that every contractive sequence is Cauchy and therefore is convergent .

Exercise 8.2: Given $x, y \in R$ and $0 < \rho < 1$, let $\{x_n\}$ be a sequence of real numbers defined by $x_1 = a , y_1 = b$ and

$$x_{n+1} = (1 + \rho)x_n - \rho\, x_{n-1} \quad \text{for all } n \geq 2 .$$

Show that $\{x_n\}$ is Cauchy Sequence and its limit is $(b + \rho a)/(1 - \rho)$.

Exercise 8.3 : Prove that the sequence $\{x_n\}$ is not a Cauchy sequence , where

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \quad \text{,for all } n \in N .$$
