SEMESTER-VI

PHYSICS-DSE: CLASSICAL DYNAMICS

SPECIAL THEORY OF RELATIVITY (PART-II)

KNU SYLLABUS- MINKOWSKI SPACE, THE INVARIANT INTERVAL, LIGHT CONE AND WORLD LINES. SPACE-TIME DIAGRAMS, FOUR VECTORS: SPACE LIKE, TIME LIKE AND LIGHT LIKE. FOUR VELOCITY AND ACCELERATION. FOUR MOMENTUM AND ENERGY MOMENTUM RELATION. CONCEPT OF FOUR FORCE.

Hermann Minkowski (1864-1909)

1) **Minkowski Space-**

Minkowski space is a combination of three-dimensional Euclidean space and time into four dimensional manifolds where the space-time interval between any two events is independent of the inertial frame of reference in which they are recorded. The four co-ordinates in Minkowski's space are (x, y, z, ict). Any point in such a space is called a world point. The motion of a particle in Minkowski's world is represented by a curve called a world line which gives the locus of the world points corresponding to the motion.

If (x_1, y_1, z_1, ict_1) and (x_2, y_2, z_2, ict_2) are the co-ordinates of any two events, then the quantity,

$S_{12} = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2]^{1/2}$ is called the interval between the two events. The intervals are called time like, space like or light like (null) according as S_{12}^2 is negative, positive or zero respectively. Let us call these regions as A, B and C such that $S_{12}^2 > 0$ in region A, $S_{12}^2 < 0$ in region B and $S_{12}^2 = 0$ on the surface C.
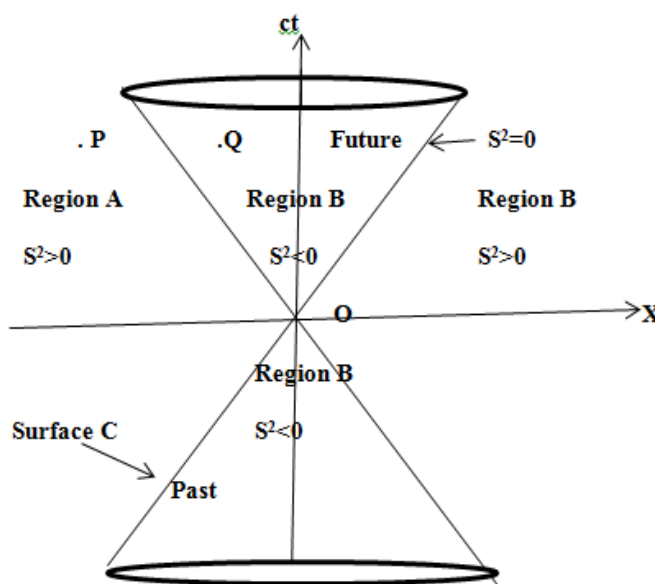

Figure No.1

Considering the two events occurring at origin and at (x, y, z, ict) then,

$$S_{12}^2 = S^2 = x^2 + y^2 + z^2 - c^2t^2 \dots\dots\dots\dots\dots\dots\dots\dots \text{ (i)}$$

Hence, the surface C is defined by,

$$S^2=0 \text{ i.e. } x^2 + y^2 + z^2 = c^2t^2 \dots\dots\dots\dots\dots\dots\dots\dots \text{ (ii)}$$

In four dimensions, equation (ii) represents the surface of a cone, called null cone, with apex at the origin and time axis ct with semi-vertical angle equal to 45^0.

In the upper cone the time co-ordinates of all events are positive in all frames of reference and, therefore, we have events which occur after the event at the origin. That is, the upper cone represents absolute

future. The lower cone similarly represents absolute past. The null cone is the space-time representation of the propagation of light. Hence, it is also called light cone.

The region A corresponds to space-like interval. For any point P in this region, we can always find a frame of reference such that the events at O and P are simultaneous. The region B corresponds to time-like interval. For any point Q in this region, we can always find a frame of reference such that the events at O and Q occur at the same place.

It can be shown that the value of S^2 is invariant to Lorentz transformations.

Lorentz transformation equations are,

$$(i)\, x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$(ii)\, y' = y$$

$$(iii)\, z' = z$$

$$(iv)\, t' = \frac{x - vx/c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

In the frame S',

$$S'^2 = x'^2 + y'^2 + z'^2 - c^2 t'^2$$

Putting the values of x', y', z' and t' in terms of x, y, z and t, we get,

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = \frac{(x - vt)^2}{\left(1 - \frac{v^2}{c^2}\right)} + y^2 + z^2 - c^2 \frac{(t - \frac{vx}{c^2})^2}{\left(1 - \frac{v^2}{c^2}\right)}$$

$$= x^2 + y^2 + z^2 - c^2 t^2$$

Thus the quantity S^2 is invariant under Lorentz transformation.

2) **Four Vector-**

If an event is represented in S-frame by (x, y, z, t) and in S' frame by (x', y', z', t'), S and S' being inertial frames, then by Lorentz transformation,

$$x^2 + y^2 + z^2 - c^2 t^2 = x'^2 + y'^2 + z'^2 - c^2 t'^2$$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2$$

where $x_1 \equiv x, x_2 \equiv y, x_3 \equiv z, x_4 \equiv ict$ and similarly for the primed co-ordinates.

Therefore, $\sum_{i=1}^{4} x_i^2 = \sum_{i=1}^{4} x_i'^2$ i.e. the quantity $\sum_{i=1}^{4} x_i^2$ is invariant under Lorentz transformation.

The result is very similar to the property of a vector in three dimensional space. Thus, we may regard (x₁, x₂, x₃, x₄) as the components of a four dimensional vector or four vector, also called the world vector. So, a four vector is a vector in four dimensional Minkowski's space.

Common four vectors are as follow-

a) **Position Four-Vector-**

It is expressed as, $x_n = (x_1, x_2, x_3, x_4) = (\vec{r}, ict) = (\vec{r}, \tau)$

b) **Velocity Four-Vector-**

The components of the velocity four vector are given by,

$$v_1 = \frac{dx_1}{d\tau} = \frac{v_x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$v_2 = \frac{dx_2}{d\tau} = \frac{v_y}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$v_3 = \frac{dx_3}{d\tau} = \frac{v_z}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$v_4 = \frac{dx_4}{d\tau} = \frac{ic}{\sqrt{1 - \frac{v^2}{c^2}}}$$

In general velocity four vector is given by,

$$v_n = \left(\frac{\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{ic}{\sqrt{1 - \frac{v^2}{c^2}}} \right), where\ \vec{v} = \frac{d\vec{r}}{d\tau}.$$

The square of magnitude of the velocity four vector is given by,

$$v_n v_n = \frac{v^2}{1-\frac{v^2}{c^2}} - \frac{c^2}{1-\frac{v^2}{c^2}} = -c^2 \ \text{which is Lorentz invariant.}$$

c) **Momentum Four-Vector-**

The components of four momentum are given by,

$$p_1 = m_0 v_1 = \frac{m_0 v_x}{\sqrt{1-\frac{v^2}{c^2}}} = m v_x = p_x$$

$$p_2 = m_0 v_2 = \frac{m_0 v_y}{\sqrt{1-\frac{v^2}{c^2}}} = m v_y = p_y$$

$$p_3 = m_0 v_3 = \frac{m_0 v_z}{\sqrt{1-\frac{v^2}{c^2}}} = m v_z = p_z$$

$$p_4 = m_0 v_4 = \frac{m_0 ic}{\sqrt{1-\frac{v^2}{c^2}}} = imc = iE/c$$

In general momentum four-vector is given by,

$$p_n = (p_x, p_y, p_z, imc) = \left(\vec{p}, \frac{iE}{c}\right) where \ \vec{p} = m\vec{v}$$

The square of the magnitude of the four momentum is given by,

$$p_n p_n = p^2 - \frac{E^2}{c^2} = -\frac{(E^2 - p^2 c^2)}{c^2} = -m_0^2 c^2 \ \text{which is Lorentz invariant.}$$

p_n is also called energy-momentum four vector.

Conservation of Four-Momentum- Let p_n^1 and p_n^2 be the four-momentum of two particles. If they collide, then according to the conservation of four momentum,

$$p_n^1 + p_n^2 = \text{Constant}$$

d) **Acceleration Four-Vector-**

Acceleration Four-Vector is given by,

$$a_n = \frac{dv_n}{d\tau} = \frac{dv_n}{dt} \cdot \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

The components of acceleration four vector are as follow-

$$a_1 = \frac{dv_1}{dt} \cdot \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \frac{d}{dt}\left(\frac{v_x}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

$$= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[\frac{\dot{v}_x}{\sqrt{1-\frac{v^2}{c^2}}} + v_x \frac{d}{dt}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)\right]$$

$$a_2 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[\frac{\dot{v}_y}{\sqrt{1-\frac{v^2}{c^2}}} + v_y \frac{d}{dt}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)\right]$$

$$a_3 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[\frac{\dot{v}_z}{\sqrt{1-\frac{v^2}{c^2}}} + v_z \frac{d}{dt}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)\right]$$

$$a_4 = \frac{dv_4}{dt} \cdot \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d}{dt}\left(\frac{ic}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{ic}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d}{dt}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

e) **Force Four-Vector-**

The Four-Force vector can be written as,

$$F_n = \frac{dp_n}{d\tau} = \frac{dp_n}{dt} \cdot \frac{dt}{d\tau} = \frac{dp_n}{dt}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

In terms of Newton's equation, force four-vector can be expressed as,

$$F_n = \frac{dp_n}{d\tau} = \frac{d}{d\tau}(m_0 v_n) = m_0 \frac{dv_n}{d\tau} = m_0 \frac{d^2 x_n}{d\tau^2}$$

The components of force four vector can be given by,

$$F_1 = \frac{dp_x}{dt} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{F_x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$F_2 = \frac{dp_y}{dt} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{F_y}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$F_3 = \frac{dp_z}{dt} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{F_z}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$F_4 = \frac{d}{dt}\left(\frac{iE}{c}\right) \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{i}{c\left(\sqrt{1 - \frac{v^2}{c^2}}\right)} \frac{dE}{dt}$$

3) **Minkowski's Space and Lorentz Transformation-**

It can be shown that the Lorentz transformation is equivalent to a simple rotation about the origin O of the orthogonal set of four axes in the four dimensional (x_1, x_2, x_3, x_4) Minkowski's world through an angle $\theta = \tan^{-1}\left(-\frac{iv}{c}\right)$.

Proof: - The square of the distance of a world point from the origin is $\sum_{i=1}^{4} x_i^2$ in S-frame and $\sum_{i=1}^{4} x_i'^2$ in S'-frame.

Lorentz transformation must leave all the distance unchanged in (x, y, z, τ) space. The required transformation can be mathematically expressed as a rotation of a four dimensional co-ordinate system.

We consider the simple case where $x_2' = x_2$ and $x_3' = x_3$ i.e. rotation in $x_1 x_4$ plane (or, x τ-plane). So, if θ be the angle of rotation of S-frame about its origin O to obtain the S'-frame with origin O' coinciding with O.
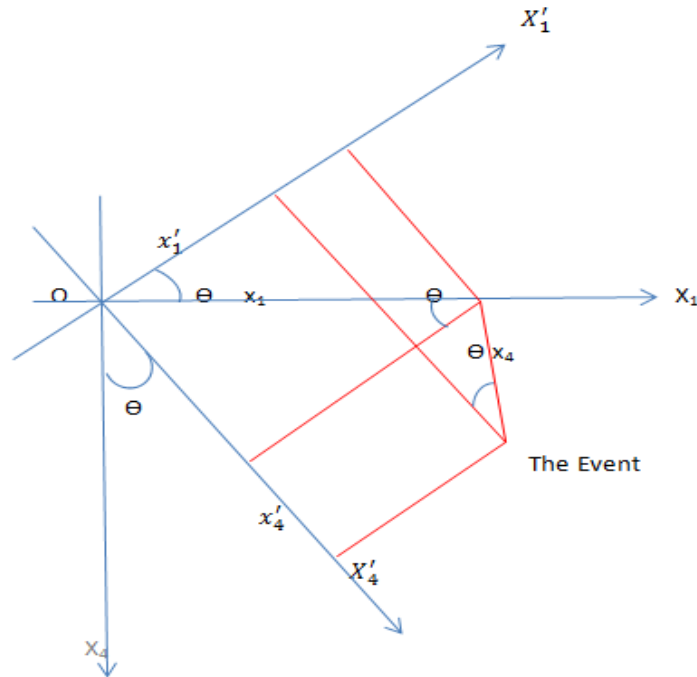
Figure No.-2

Thus, from the figure we can write,

$$\begin{pmatrix} x_1' \\ x_4' \end{pmatrix} = \begin{pmatrix} Cos\theta & -Sin\theta \\ Sin\theta & Cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}$$

Thus,

$$x_1' = x_1 Cos\theta - x_4 Sin\theta \dots\dots\dots\dots\dots \text{ (i)}$$

$$x_4' = x_1 Sin\theta + x_4 Cos\theta \dots\dots\dots\dots\dots \text{ (ii)}$$

Let us consider the motion, in S-frame, of the origin of the S'-frame.

Therefore, $x_1'=0$. Also, we have $x_1 = vt$.

Now, equation (i) gives, $x_1 Cos\theta - x_4 Sin\theta = 0$

So, $\tan\theta = \dfrac{x_1}{x_4} = \dfrac{vt}{ict} = -i\beta, \left(as\ \beta = \dfrac{v}{c} \right)$

and $Sin\theta = -\dfrac{i\beta}{\sqrt{1-\beta^2}}; Cos\theta = \dfrac{1}{\sqrt{1-\beta^2}}.$

Putting the values of SinΘ and CosΘ in equations (i) and (ii), we get

$$x_1' = \frac{x_1}{\sqrt{1-\beta^2}} + \frac{x_4 i\beta}{\sqrt{1-\beta^2}} = \frac{x_1 - vt}{\sqrt{1-\beta^2}}. \quad (as\, x_4 \equiv ict)$$

$$x_4' = \frac{-x_1 i\beta}{\sqrt{1-\beta^2}} + \frac{x_4}{\sqrt{1-\beta^2}} \quad or, t' = \frac{t - \dfrac{vx_1}{c^2}}{\sqrt{1-\beta^2}} \quad \left(as\, \frac{x_4'}{ic} = t'\right)$$

Thus, we get the following equations,

i) $\quad x_1' = \dfrac{x_1 - vt}{\sqrt{1-\beta^2}}$

ii) $\quad x_2' = x_2$

iii) $\quad x_3' = x_3$

iv) $\quad t' = \dfrac{t - \frac{vx_1}{c^2}}{\sqrt{1-\beta^2}}$

These are the Lorentz transformation equations.

Exercise-

1. What is Minkowski's space? Show that the Lorentz transformation may be regarded as rotation of axes through an imaginary angle

$$\theta = \tan^{-1}(i\beta) \; where \; \beta = \frac{v}{c}.$$

2. How is four-vector defined in relativity? Write down the components of the momentum four-vector in any Lorentz frame. Find scalar product of this four-vector with itself.

3. Explain how Minkowski's four-dimensional space-time description of events and intervals is consistent with the postulates of special theory of relativity.

4. State Lorentz transformations in four-dimensional space representation.

5. What is meant by space-time intervals or separation between two events? When is the separation said to be space-like and when is it time-like? Discuss the time order of two events in the two cases of separation.

6. Discuss Minkowski's four-dimensional space-time continuum.

7. Show that the Lorentz transformation can be regarded as transformation due to rotation of axes in the four-dimensional Minkowski's space.

8. State the law of conservation of four-momentum. Show that the square of the magnitude of the four-momentum is Lorentz invariant.

9. Show that the square of the magnitude of the velocity four vector is Lorentz invariant.

References-

1) S. P. Kuila, Concepts of Atomic Physics, New Central Agency (P) Ltd (2018).
2) R. Resnick, Introduction to Special Theory of Relativity, New York, Weley (1968).
3) A. Beiser, Perspective of Modern Physics, 3rd Edition, New York, McGraw Hill (1969).

AJAY KUMAR SHARMA
Assistant Professor (Physics)
B.C.College, Asansol
Email-id- ajay888sharma@yahoo.co.in