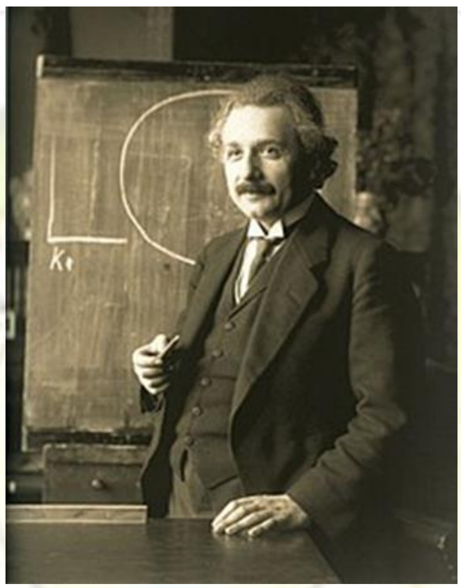
SEMESTER-VI

PHYSICS-DSE: CLASSICAL DYNAMICS

SPECIAL THEORY OF RELATIVITY (PART-I)

KNU SYLLABUS-

POSTULATES OF SPECIAL THEORY OF RELATIVITY, LORENTZ TRANSFORMATIONS, TIME-DILATION, LENGTH CONTRACTION, TWIN PARADOX



ALBERT EINSTEIN
(1879-1955)

Basic Concepts-

a) A system of co-ordinate axes which defines the position of a particle in two or three dimensional space is called a frame of reference. Frame of references are of two types- Inertial and Non-inertial frames.

b) Inertial frames are un-accelerated frames while Non-inertial frames are accelerated frames.

c) According to Newtonian principle of relativity- "Absolute motion, which is the transition of a body from one absolute place to another absolute place, can never be detected. Translatory motion can be perceived only in the form of motion relative to other material bodies"

d) The Galilean Transformation Equations are-

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

e) In 1887, Michelson and Morley by their experiment proves that there is no ether like medium.

Contents-

1. ## Postulates of Special Theory of Relativity-

Einstein proposed the special theory of relativity in 1905. The basic postulates of this theory are as follow-

a) **The laws of physics are the same in all inertial frames of reference.**

b) **The velocity of light in free space is constant. It is independent of the relative motion of the source and the observer.**

2. ## Lorentz Transformation-

The equations in relativity physics which relate the space and time co-ordinates of two co-ordinate systems moving with a uniform velocity relative to one another are called Lorentz Transformation.

Consider two observers O and O' in two systems S and S'. System S'- is moving with a constant velocity *v* relative to system S along the positive X-axis.

Suppose we make measurements of time from origins of S and S'- just coincide i.e. t=0 when O and O' coincide.

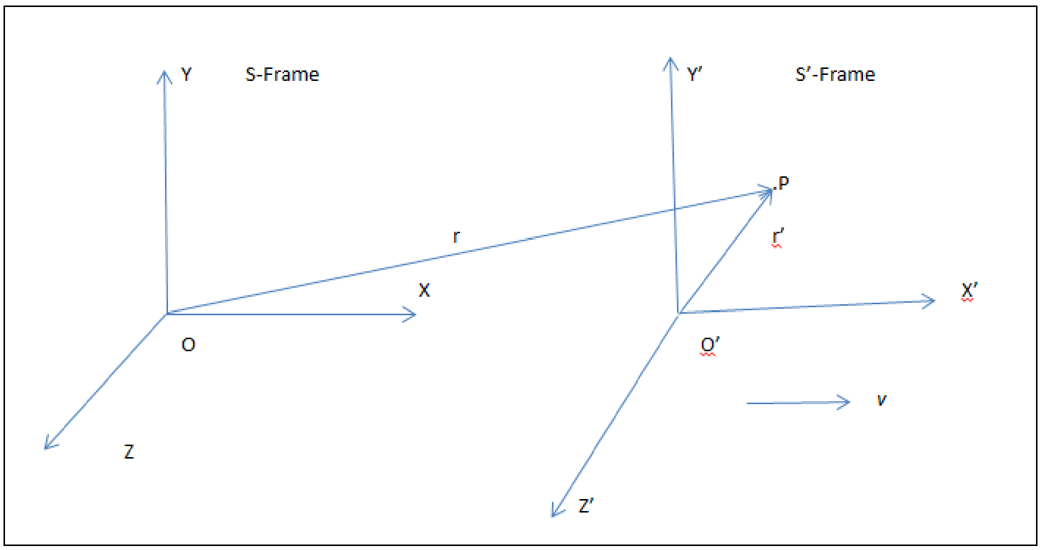


Figure No.-1

Suppose a light pulse is emitted when O and O' coincide. The light pulse produced at t=0 will spread out as a growing sphere. The radius of the wave-front produced in this way will grow with speed c.

After a time t, the observer O will note that the light has reached a point P(x,y,z) as shown in figure. For him, the distance of the point P is given by, r=ct.

From figure,

$$r^2 = x^2 + y^2 + z^2$$

$$x^2 + y^2 + z^2 = c^2t^2 \text{ .... (i)}$$

Similarly, the observer O' will note that the light has reached the same point P in a time t' with the same velocity c. So, r'=ct'.

So from the figure we can write,

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \text{ .... (ii)}$$

Now, equation (i) and (ii) must be equal since both the observers are at the centre of the same expanding wave-front.

Thus, $x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c'^2t'^2$....................................... (iii)

Since there is no motion in the Y and Z- directions, y'=y and z'=z.

So, equation (iii) becomes,

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \text{ .... (iv)}$$

The transformation equation relating to x and x' can be written as,

$$x' = k(x - vt) \text{ .... (v)}$$

where k is a constant.

The reason for trying the above relation is that, the transformation must reduce to Galilean transformation for low speed (v<<c).

Similarly, let us assume that,

$$t' = a(t - bx) \text{ .... (vi)}$$

where a and b are constants.

Substituting these values for x' and t' in equation (iv), we have,

$$x^2 - c^2t^2 = k^2(x - vt)^2 - c^2a^2(t - bx)^2$$

$$x^2 - c^2t^2 = k^2(x^2 - 2xvt + v^2t^2) - c^2a^2(t^2 - 2bxt + b^2x^2)$$

$$x^2 - c^2t^2 = (k^2 - c^2a^2b^2)x^2 - 2(k^2v - c^2a^2b)xt - (a^2 - \frac{k^2v^2}{c^2})c^2t^2 \text{ .... (vii)}$$

Equating the coefficients of x^2, c^2t^2 we get,

$$k^2 - c^2a^2b^2 = 1 \text{ .... (viii)}$$

$$k^2v - c^2a^2b = 0 \text{ .... (ix)}$$

$$a^2 - \frac{k^2v^2}{c^2} = 1 \text{...(x)}$$

Solving the above equations for k, a and b, we get,

$$k = a = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$b = v/c^2$$

Putting the values of k, a and b in equation (v) and (vi) we get,

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Therefore, the Lorentz transformation equations are-

$$(i) x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$(ii)\ y' = y$$

$$(iii)\ z' = z$$

$$(iv)\ t' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The inverse Lorentz transformation equations are obtained by interchanging the co-ordinates and replacing *v* by –*v* in the above equations-

$$(i)\ x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$(ii)\ y = y'$$

$$(iii)\ z = z'$$

$$(iv)\ t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

These equations convert measurements made in frame S' into those in frame S.

Special Case:-

When v<<c, then

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1$$

Hence Lorentz transformation equations reduces to,

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

These are Galilean transformations.

3. Consequences of Lorentz transformation equations-

a) Length Contraction-

Let us consider two systems S and S' with their X-axes coinciding at time t=0. Let S' is moving with a uniform relative speed *v* with respect to S in the positive X-direction. Consider a rod PQ, at rest relative to S'.

Let x_1' and x_2' be the co-ordinates of the ends of the rod at any instant of time in S'.

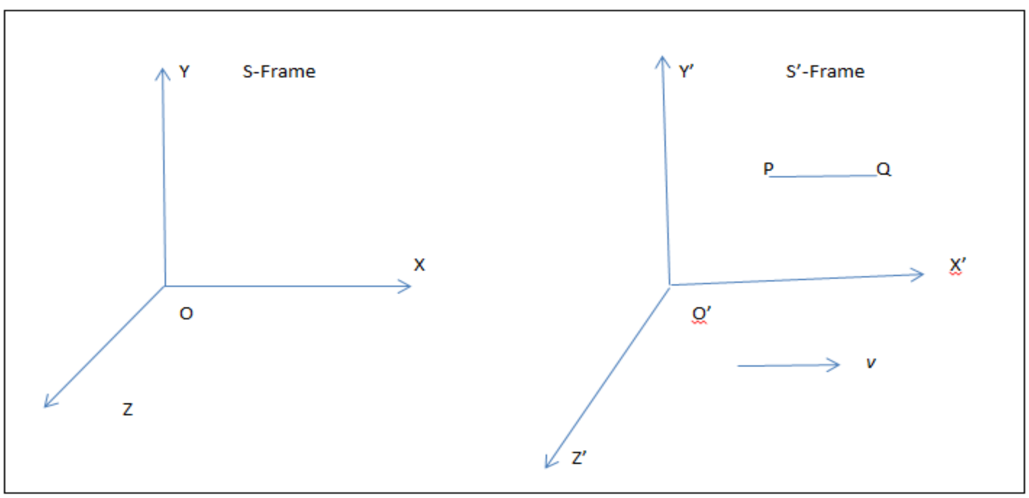


Figure No.-2

Then, length of the rod measured relative to S' is,

$$l_o = x_2' - x_1' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Similarly, let x_1 and x_2 be the co-ordinates of the ends of the rod at the same instant of time in S.

Then, length of the rod measured relative to S is,

$$l = x_2 - x_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Now, according to Lorentz transformations,

$$x_1' = \frac{x_1 - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

$$and\ x_2' = \frac{x_2 - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

So, equation (i) gives,

$$l_o = x_2' - x_1' = \frac{x_2 - vt}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{x_1 - vt}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{x_2 - x_1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{l}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$or, l = l_o\sqrt{1 - \frac{v^2}{c^2}} \ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}$$

From equation (v) it is clear that $l < l_o$. Therefore, to the observer in S it would appear that the length of the rod (in S') has contracted by the factor $\sqrt{1 - \frac{v^2}{c^2}}$.

<u>Special Cases:</u>

i) The proper length of an object is the length determined by an observer at rest with respect to the object. In the above case, l_o is the proper length.

ii) The shortening or contraction in the length of an object along its direction of motion is known as the Lorentz-Fitzgerald contraction.

iii) There is no contraction in a direction perpendicular to the direction of motion.

iv) A body which appears to be spherical to an observer at rest relative to it, will appear to be an oblate spheroid to a moving observer. Similarly, a square and a circle in one appear to the observer in the other to be a rectangle and an ellipse respectively.

v) The contraction is reciprocal i.e. if two identical rods are at rest- one in S and the other in S', each of the observers find that the other is shorter than the rod of his own system.

b) <u>Time Dilation-</u>

Let us consider two systems S and S' with their X-axes coinciding at time t=0. Let S' is moving with a uniform relative speed v with respect to S in the positive X-direction.

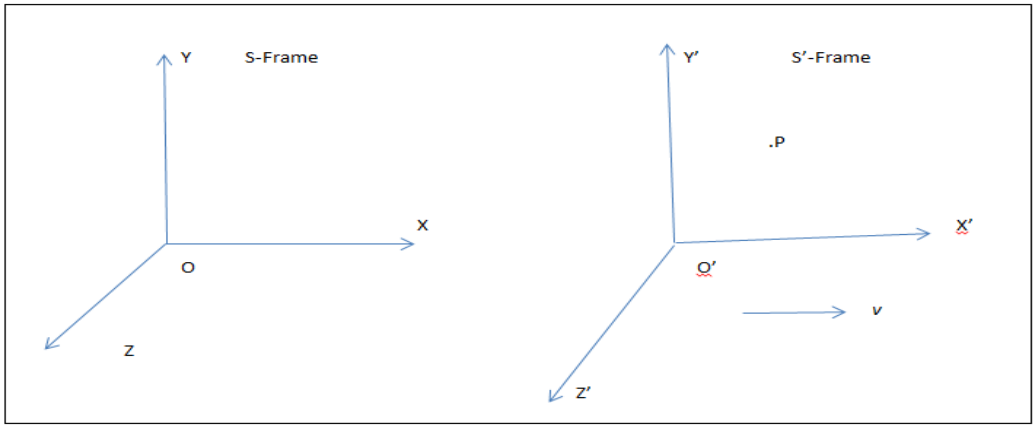


Figure No.-3

Suppose a gun placed at the position P (x',y',z') in S' fires two shots at times t_1' and t_2' measured with respect to S'. In S' the clock is at rest relative to the observer. The time interval measured by a clock at rest relative to the observer is called the proper time interval. Hence, the time interval between the two shots for the observer in S' i.e. proper time interval is, $t_o = t_2' - t_1'$.

Since, the gun is fixed in S', it has a velocity *v* with respect to S in the direction of the positive X-axis.

Let $t = t_2 - t_1$ represent the time-interval between the two shots as measured by an observer in S.

From inverse Lorentz transformations, we can write,

$$t_1 = \frac{t_1' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$and\ \ t_2 = \frac{t_2' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\text{So, } t_2 - t_1 = \frac{t_2' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{t_1' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$t = \frac{t_2' - t_1'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$t = \frac{t_o}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{.......................................(i)}$$

From equation (i), it is clear that

$$t > t_o$$

Thus, the time-interval, between two events occurring at a given point in the moving frame S' appear to be longer to the observer in the stationary frame S i.e. a stationary clock measures a longer time interval between events occurring in a moving frame of reference than does a clock in the moving frame. This effect is called time-dilation .

Special Cases-

i) **Twin Paradox- Consider two exactly identical twin brothers. Let one of the twins go to a long space journey at a high speed in a rocket and the other stay behind on the earth. The clock in the moving rocket will appear to go slower than the clock on the surface of the earth, in accordance with the relation,**

$$t = \frac{t_o}{\sqrt{1-\frac{v^2}{c^2}}}$$

Therefore, when he returns back to the earth, he will find himself younger than the twin who stayed behind on the earth.

ii) **The faster a frame of reference moves, the slower the clock in it shall appear to run. If $v=c$, $t=\infty$ i.e. a clock moving with velocity of light relative to an observer, appears to stop running completely.**

Solved Examples

1) **The length of a rod is found to be half of its length when at rest. What is the speed of the rod relative to the observer?**

Solution-

Consider l_o be the length of the rod when it is in rest and l be its length when it is moving with velocity v.

Here, $l = l_o/2$

Now using the length contraction formula, $l = l_o\sqrt{1 - \frac{v^2}{c^2}}$ we get,

$$\frac{1}{2}l_o = l_o\sqrt{1-\frac{v^2}{c^2}}$$

On solving we get, *v =0.866c.*

2) **Half-life of a particle at rest is 20 nanoseconds. What will be the half-life if its speed is 0.9c?**

Solution-

Let half-life of the particle when it is in rest condition is t_o *i.e.*t_o= 20x10^{-9}Sec.

Let *t* be the half-life of the particle when its speed is *v=0.9c.*

Using the relation, $t = \frac{t_o}{\sqrt{1-\frac{v^2}{c^2}}}$ we can write,

$$t = \frac{20x10^{-9}}{\sqrt{1-\frac{0.81c^2}{c^2}}} = 45.5x10^{-9}Sec.$$

3) **Show that the circle $x^2 + y^2 = a^2$ in frame-S appears to be an ellipse in frame S' which is moving with a velocity *v* with respect to S.**

Solution-

Equation of a circle in frame S is given by,

$x^2 + y^2 = a^2$……………………….. (i)

According to length contraction relation, $x = x'\sqrt{1-\frac{v^2}{c^2}}$

y=y'

So, equation (i) becomes,

$$x'^2\left(1-\frac{v^2}{c^2}\right) + y'^2 = a^2$$

$$or, \frac{x'^2}{a^2}\left(1-\frac{v^2}{c^2}\right) + \frac{y'^2}{a^2} = 1$$

This is the equation of an ellipse.

Exercise-

Numerical Problems

1) **A rod 1m long is moving along its length with a velocity 0.6c. Calculate its length as it appears to an observer (a) on the earth (b) moving with the rod itself.**
2) **How fast would a rocket have to go relative to an observer for its length to be contracted to 99% of its length at rest?**

3) At what speed should a clock be moved so that it may appear to lose 1 minute in each hour?

4) Calculate the percentage contraction of a rod moving with a velocity 0.8c in a direction inclined at 60^0 to its own length.

5) Show that if (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) are the co-ordinates of one event in S_1 and the corresponding event in S_2 respectively, then the expression $ds_1^2 = dx_1^2 + dy_1^2 + dz_1^2 - c^2 dt_1^2$ is invariant under Lorentz transformation of co-ordinates.

6) Show by direct application of Lorentz transformation, $x'^2 - c^2 t'^2 = x^2 - c^2 t^2$.

7) Show that the four dimensional volume dx dy dz dt is invariant under Lorentz transformation.

8) How much younger an astronaut will appear to the earth observer, if he returns after one year having moved with velocity 0.5c?

Theoretical Problems-

1) State the basic postulates of special theory of relativity.
2) What do you mean by proper length and proper time?
3) Derive Lorentz transformation equations based on the fundamental postulates of special theory of relativity. Show that for small velocities they reduce to Galilean transformation.
4) Using Lorentz transformation explain –i) length contraction and ii) time dilation.
5) Explain twin paradox using the concept of time dilation.
6) Explain the phenomenon of Lorentz-Fitzgerald contraction.

References-

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