PHYSICS- DSE : APPLIED OPTICS

CHAPTER : MATRIX METHOD

Matrix : It is a rectangular arrangement of elements in m rows and n columns, where m and n are natural numbers. This is called a $m \times n$ matrix, and has $m \times n$ elements.

If we consider a matrix,

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

it has 2 rows and 3 columns and the number of elements is $2 \times 3 = 6$. This matrix is known as 'rectangular matrix'.

Another matrix of the type. $B = \begin{bmatrix} g \\ h \\ i \end{bmatrix}$ – has only one column and is known as 'column matrix'.

A matrix having 2 rows and 2 columns is called a 'square matrix'.

Two matrices Multiplication rule :-

(i) Multiply the elements of the first row of matrix A by the corresponding elements of the first column of the matrix B and add. This will give the first element of the first row of the matrix C.

(ii) Multiply the elements of the first row of A by the corresponding elements of the second column of B and add. This will give the second element of the first row of C.

Proceeding as above, we can get the elements of the second row of C, as will be clear from the following example.

Let, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then

$$C = AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{pmatrix} [ae+bg] & [af+bh] \\ [ce+dg] & [cf+dh] \end{pmatrix}$$

Matrix formation :-

Consider a set of two equations

$$x_1 = ay_1 + by_2 \; ; \; x_2 = cy_1 + dy_2$$

considering the multiplication rule, we can put these equations in the form,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{.....(i)}$$

further again if we have,

$$y_1 = ez_1 + fz_2 \; , \; y_2 = gz_1 + hz_1 \; , \text{ then.}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \text{........ (ii)}$$

From (i) and (ii), we get,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \text{....... (iii)}$$

or, $\quad X = BZ$

where, X and Z represents (2×1) matrices and B represents (2×2) square matrix. Thus.

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$= \begin{pmatrix} [ae + bg] & [af + bh] \\ [ce + dg] & [cf + dh] \end{pmatrix} \quad \text{.....(iv)}$$

comparing (iii) & (iv), we get,

$$x_1 = (ae + bg)z_1 + (af + bh)z_2$$

and, $$x_2 = (ce + dg)z_1 + (cf + dh)z_2$$

Matrix Method :-

In Optics, a system may consist of a number of mirrors and lenses or their combinations. In order to find the nature and position of the final image, we have to proceed step by step using a ray diagram and applying the laws of reflection and refraction. This process is generally quite difficult and laborious and becomes highly complicated if the number of mirrors and lenses is very large.

It has been found that matrix algebra can prove a very powerful mathematical tool for this purpose. The matrices can easily be related to the process of reflection or refraction by describing the rays in terms of their height above the principal axis and the angle the ray makes with the axis. These quantities are termed as co-ordinates of the ray.

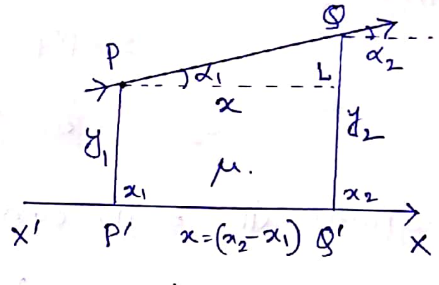
The method of writing linear equations for the co-ordinates of the paraxial rays and then transforming them into matrices for the purpose of optical analysis is called 'matrix method' in optics.

Why suitable for paraxial rays : Matrix method is suitable for paraxial rays i.e. for the rays which make very small angles with the principal axis because the changes in co-ordinates of the rays in such a case can be expressed as linear equation which can then be analysed through the method of matrices.

Co-ordinates of a ray: Consider a cylindrically symmetrical optical system having x-axis as the axis of the system. The paraxial rays which pass through the axis of the system are confined to a single plane (XY plane).

A ray can be specified by its distance from the axis of the system and the angle that it makes with the axis.

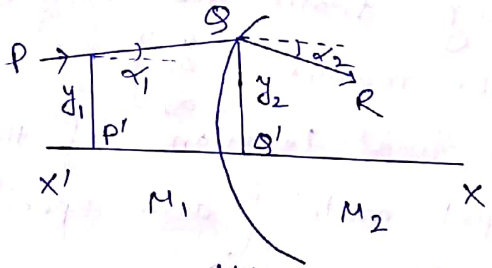
Take a ray PQ inclined to the axis XX' (in fig: (a)) consider a point P at a distance y_1 from the axis. The ray makes an angle α_1 with the axis at P. Then the quantities (y_1, α_1) represents the co-ordinates of the ray at P. Similarly, the quantities (y_2, α_2) represents the co-ordinates of the ray at Q.



(a)

Here, $\alpha_2 = \alpha_1$

In figure: (b) the ray travels from a medium of refractive index μ_1 to a medium of refractive index μ_2. It strikes the refracting surface at Q and converges along QR making an angle α_2 with the direction of principal axis. We, therefore, have seperate co-ordinates for the input ray PQ and output ray QR.



(b)

The input ray PQ makes an angle α_1 with the axis whereas output ray makes an angle α_2. The distance y_2 of the point Q from the axis is the same for the input ray as well as the output ray. Therefore, (y_2, α_1) are the co-ordinates of the point Q ~~from~~ for the input ray and (y_2, α_2) for the output ray.

Effect of Translation :- Consider a ray PQ (fig: (a)) travelling in a homogeneous medium of refractive index μ. Let, (y_1, α_1) and (y_2, α_2) be the co-ordinates of the ray at P and Q respectively.

As the medium is homogeneous the ray travels in a straight line. Hence, $\alpha_1 = \alpha_2$ ---(i)

Now, $PP' = y_1$ & $QQ' = y_2$ are perpendiculars on the axis. and $P'Q' = (x_2 - x_1) = x$.

As the ray is paraxial α_1 is very small so that

$$\tan\alpha_1 \simeq \alpha_1$$

Also,

$$QQ' = Q'L + LQ.$$

$$= P'P + x\tan\alpha_1$$

$$\therefore\ y_2 = y_1 + x\alpha_1 \quad \text{----(ii)}$$

Equations (ii) & (i) can be written as,

$$y_2 = (1)\, y_1 + (x)\, \alpha_1 \quad \text{----(iii)}$$

and

$$\alpha_2 = (0)\, y_1 + (1)\, \alpha_1 \quad \text{----(iv)}.$$

Equations (iii) & (iv) can be combined into the following matrix equation.

$$\begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix} \quad \text{---(v)}.$$

Matrix equation (v) represents the translation of the ray from the co-ordinates (y_1, α_1) to (y_2, α_2).

The matrix $\begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix}$ is called output matrix and specifies the final location of the ray. The matrix $\begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix}$ is called input matrix and specifies initial location of the ray.

The matrix

$$T_m = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

- is called the Translation Matrix. represents the translation operation.

The determinant of translation matrix,

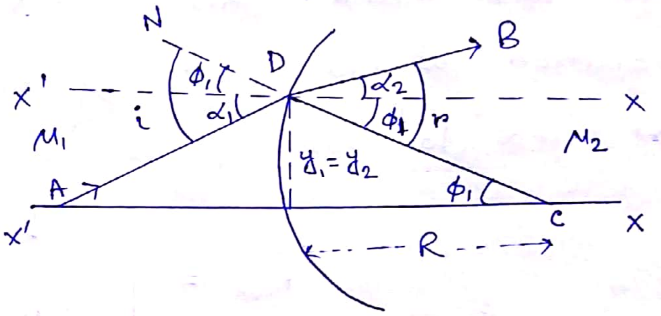
$$|T_m| = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1.$$

Refraction Matrix

Consider a convex spherical refracting surface of radius of curvature R seperating the two media of refractive index μ_1 and μ_2 such that $\mu_2 > \mu_1$. A ray AD is incident on the spherical surface at D making an angle of incident i with the normal CDN.

It is refracted along DB making an angle of refraction r with the normal. According to Snell's law

$$\mu_1 \sin i = \mu_2 \sin r$$



For paraxial rays both i and r are small.

$$\therefore \sin i = i, \sin r \simeq r.$$

Hence, $\mu_1 i = \mu_2 r$

If ϕ_1 is the angle which the normal makes with the X-axis and α_1 and α_2 the angles with the incident ray and refracted ray make with the direction of x-axis.

then, $i = \phi_1 + \alpha_1$ & $r = \phi_1 + \alpha_2$

As, ϕ_1 is small, $\phi_1 = y_1/R$.

$$\therefore \mu_1(\phi_1 + \alpha_1) = \mu_2(\phi_1 + \alpha_2)$$

or, $$\mu_2\alpha_2 = \mu_1\alpha_1 + \phi_1(\mu_1 - \mu_2)$$

$$= \mu_1\alpha_1 + \frac{\mu_1 - \mu_2}{R} y_1$$

$$\therefore \alpha_2 = \frac{y_1}{R}\left(\frac{\mu_1}{\mu_2} - 1\right) + \frac{\mu_1}{\mu_2}\alpha_1 \quad \text{---(i)}$$

As the height of the ray before and after refraction same

$$y_2 = y_1 \quad \text{-----(ii)}$$

Equation (ii) can be written as,

$$y_2 = (1)y_1 + (0)\alpha_2 \quad \text{----(iii)}$$

The set of equation (i) and (iii) can be put in the matrix form as under,

$$\begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{\mu_1}{\mu_2}-1\right) & \frac{\mu_1}{\mu_2} \end{bmatrix} \begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix}$$

This matrix equation represents the refraction of the ray at the point D on the convex surface seperating the two media. This matrix $\begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix}$ gives the initial location of the input ray and matrix $\begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix}$ gives the final location of the output ray. The matrix.

$$R_m = \begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{\mu_1}{\mu_2}-1\right) & \frac{\mu_1}{\mu_2} \end{bmatrix}$$

representing the refraction operation of the ray is called refraction matrix.

The determinant of refraction matrix

$$|R_m| = \begin{vmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{\mu_1}{\mu_2}-1\right) & \frac{\mu_1}{\mu_2} \end{vmatrix} = \frac{\mu_1}{\mu_2}$$

The ~~matrix~~ term $\frac{1}{R}\left(\frac{\mu_1}{\mu_2}-1\right)$ in the refraction matrix, is generally written as $-P$ where P is the power defined as the reciprocal of the focal length. Therefore, in terms of power, refraction matrix is written as,

$$R_m = \begin{bmatrix} 1 & 0 \\ -P & \frac{\mu_1}{\mu_2} \end{bmatrix}$$

Thus, the refraction through a spherical surface can be characterised by a (2×2) matrix given above,

For a plane surface $R=\infty$ $\therefore \frac{1}{R}=0$ and hence $P=0$

$\therefore$ For a plane surface refraction matrix

$$R_{mp} = \begin{vmatrix} 1 & 0 \\ 0 & \frac{\mu_1}{\mu_2} \end{vmatrix}$$

<u>Focal length</u> :- As, $P = \frac{1}{R}\left(\frac{\mu_1}{\mu_2} - 1\right)$

$$= \frac{1}{R}\left(\frac{\mu_2 - \mu_1}{\mu_2}\right)$$

$$\therefore \quad \frac{1}{f} = \frac{1}{R}\left(\frac{\mu_2 - \mu_1}{\mu_2}\right)$$

$$\text{or,} \quad f = \frac{\mu_2 R}{\mu_2 - \mu_1}$$

If, $\mu_1 = 1$ and $\mu_1 = \mu_2$ then $P = \frac{\mu - 1}{\mu R}$

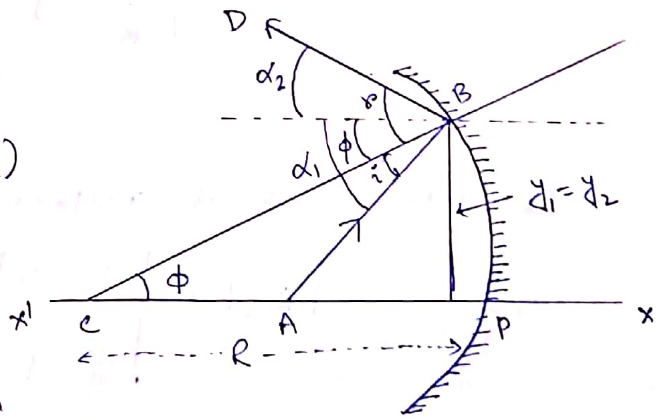
and $f = \frac{\mu R}{\mu - 1}$

(i) If the surface is convex, R is positive.

(ii) If the surface is concave, R is negative.

<u>Reflection matrix</u> :-

Consider a ray AB incident on a reflecting surface at an angle of incidence i. BC is the normal to the surface, c being the centre of curvature BD is the reflected ray which makes an angle of reflection r with the normal.



ϕ is the angle that the normal makes with the principal axis. Let, (y_1, α_1) be the input co-ordinates of the incident ray AB and (y_2, α_2) the output co-ordinates of the reflected ray BD at the point of reflection B.

Now, $y_2 = y_1$ ------(i)

and $\alpha_1 = i + \phi$ or $i = \alpha_1 - \phi$.

$\alpha_2 = r - \phi$ or $r = \alpha_2 + \phi$.

As, ϕ is very small, $\phi = \tan\phi = -\, y_1/R$.
as R is negative according to the sign convention

$$\therefore\; i = \frac{y_1}{R} + \alpha_1 \quad \text{and} \quad r = -\, y_1/R + \alpha_2$$

According to laws of reflection $i = r$.

$$\therefore\; y_1/R + \alpha_1 = -\, y_1/R + \alpha_2$$

$$\therefore\; \alpha_2 = \frac{2y_1}{R} + \alpha_1 \qquad \text{------ (ii)}$$

Equation (i) and (ii) can be written as,

$$y_2 = (1)\, y_1 + (0)\, \alpha_1 \qquad \text{------ (iii)}$$

and $$\alpha_2 = \left(\frac{2}{R}\right) y_1 + (1)\, \alpha_1 \qquad \text{--- (iv)}.$$

Writing equations (iii) & (iv) in the matrix form we get,

$$\begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix}$$

This matrix equation represents the reflection of a ray at a spherical surface $\begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix}$ represents the final location of the ray as output of reflection, $\begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix}$ represents the initial location of the ray as input for reflection and

$$R_m' = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$$

gives the reflection matrix which specifies the reflection operation.

The determinant of reflection matrix,

$$|R_m'| = \begin{vmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{vmatrix} = 1$$

System Matrix :-

An optical system, in general is made up of a series of lenses and thus can be characterised by refraction and translational matrices.

The (2×2) matrix by which the (2×1) matrix representing the orientation of the incoming incident ray may be multiplied to obtain the (2×1) matrix representing the orientation of the outgoing ray is called the system matrix.

It is denoted as,

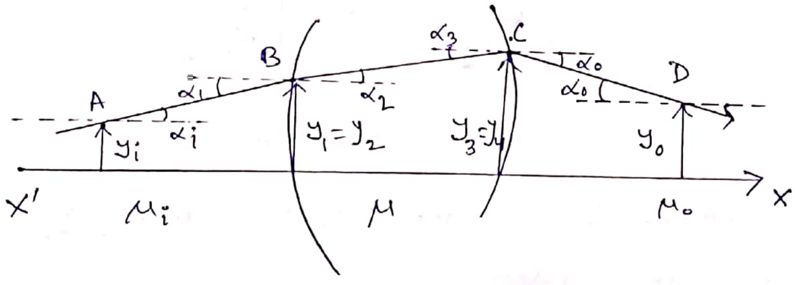
$$S_m = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The matrix equation is written as,

$$\begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix} = S_m \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix}$$

Output Matrix = System Matrix × Input Matrix.

Consider a refracting surface B seperating two media of refractive index μ_i and μ and a refracting surface C again seperating two media of refractive index μ and μ_0 forming an optical system.



A ray of light AB travelling in a homogeneous medium of refractive index μ_i meets the refracting surface at B. Where it suffers refraction, travels along BC. in a medium of refractive index μ and emerges as the ray CD into the medium of refractive index μ_0 after refraction at C.

(i) The point A is the input point. It has co-ordinates (y_i, α_i). The point B has the input co-ordinates (y_1, α_1). The translation matrix T_{m_1} from A to B gives, $\begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix} = T_{m_1} \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix}$ ---(i)

(ii) At the point B refraction takes place. The input co-ordinates are (y_1, α_1) and output co-ordinates (y_2, α_2). The refraction matrix R_{m1} at B gives,

$$\begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix} = R_{m1} \begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix} \quad \text{---- (ii)}$$

(iii) Again, there is translation from B to C, the point B has the output co-ordinates (y_2, α_2) and the point C input co-ordinates (y_3, α_3). The translation matrix T_{m2} between B & C gives,

$$\begin{bmatrix} y_3 \\ \alpha_3 \end{bmatrix} = T_{m2} \begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix} \quad \text{----- (iii)}$$

(iv) At the point C again refraction takes place. The input co-ordinates are (y_3, α_3) and output co-ordinates (y_4, α_0). The refraction matrix R_{m2} gives,

$$\begin{bmatrix} y_4 \\ \alpha_0 \end{bmatrix} = R_{m2} \begin{bmatrix} y_3 \\ \alpha_3 \end{bmatrix} \quad \text{---- (iv).}$$

(v) Again there is translation from C to D. The input co-ordinates at C are (y_4, α_0) and output co-ordinates are (y_0, α_0). The translation matrix T_{m3} between C and D gives,

$$\begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix} = T_{m3} \begin{bmatrix} y_4 \\ \alpha_0 \end{bmatrix} \quad \text{---- (v)}$$

Multiplying from (i) to (v), we have,

$$\begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix} = T_{m3} \times R_{m2} \times T_{m2} \times R_{m1} \times T_{m1} \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix}$$

$$\therefore \begin{bmatrix} y_0 \\ \alpha_0 \end{bmatrix} = S_m \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix}$$

$$\therefore O_m = S_m I_m$$

where O_m is the output matrix, I_m the input matrix, and S_m the system matrix.

$$S_m = T_{m3} \times R_{m2} \times T_{m2} \times R_{m1} \times T_{m1}$$

To get the system matrix for the whole optical system various step-wise matrices are to be multiplied in the reverse order starting from the last and finally reaching the first matrix. The order has to be strictly maintained.

Determinant $|S_m| = |T_{m3}| \times |R_{m2}| \times |T_{m2}| \times |R_{m1}| \times |T_{m1}|$

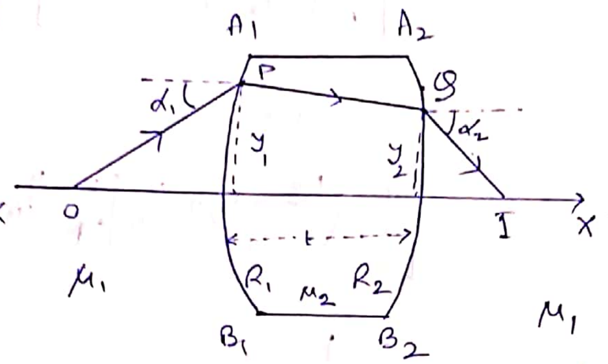
$$= 1 \times \frac{\mu}{\mu_0} \times 1 \times \frac{\mu_i}{\mu} \times 1 = \frac{\mu_i}{\mu_0}$$

In general, $S_m = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $|S_m| = ad - bc = \frac{\mu_i}{\mu_o}$.

System Matrix for Thick lens & Thin Lens

Consider a thick lens of thickness t and bounded by two spherical surfaces A_1B_1 and A_2B_2 of radii of curvature R_1 and R_2 respectively. Let, μ_2 be the refractive index of the material of the lens. and μ_1 that of the medium in which the lens is placed.

A ray of light starting from the point O on the principal axis of the lens strikes the surface A_1B_1 at P and after refraction goes from P to Q where it again suffers a second refraction and finally emerges giving rise to the image at I. Thus, in addition to the two refractions at P and Q, the ray suffers a translation through the distance t in the lens. Therefore, the matrix for thick lens may be written as,



$$S_{m(thick)} = R_{m2} \times T_m \times R_{m1}$$

where R_{m2} is the refraction matrix for 2nd refraction at Q, T_m the translation matrix from P to Q and R_{m1} the refraction matrix for first refraction at P.

$$\therefore S_{m(thick)} = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_2}\left(\frac{\mu_1}{\mu_2}-1\right) & \frac{\mu_2}{\mu_1} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{R_1}\left(\frac{\mu_1}{\mu_2}-1\right) & \frac{\mu_1}{\mu_2} \end{bmatrix}$$

We can find the value of system matrix for a thick lens when t is given

System matrix for a thin lens at $t=0$ is,

$$S_{m(thin)} = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_2}\left(\frac{\mu_2}{\mu_1}-1\right) & \frac{\mu_2}{\mu_1} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ \frac{1}{R_1}\left(\frac{\mu_1}{\mu_2}-1\right) & \frac{\mu_1}{\mu_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \frac{1}{R_2}\left(\frac{\mu_1}{\mu_2}-1\right) & \frac{\mu_2}{\mu_1} \end{bmatrix} \begin{bmatrix} 1+0 & 0+0 \\ 0+\frac{1}{R_1}\left(\frac{\mu_1}{\mu_2}-1\right) & 0+\frac{\mu_1}{\mu_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \frac{1}{R_2}\left(\frac{\mu_1}{\mu_2}-1\right) & \frac{\mu_2}{\mu_1} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ \frac{1}{R_1}\left(\frac{\mu_1}{\mu_2}-1\right) & \frac{\mu_1}{\mu_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 0+0 \\ \frac{1}{R_2}\left(\frac{\mu_2}{\mu_1}-1\right)+\frac{\mu_2}{\mu_1}\left[\frac{1}{R_1}\left(\frac{\mu_1}{\mu_2}-1\right)\right] & 0+\frac{\mu_2}{\mu_1}\times\frac{\mu_1}{\mu_2} \end{bmatrix}$$

$$S_{m(thin)} = \begin{bmatrix} 1 & 0 \\ \left(\frac{\mu_2-\mu_1}{\mu_1}\right)\left[\frac{1}{R_2}-\frac{1}{R_1}\right] & 1 \end{bmatrix}$$

The quantity $\left(\frac{\mu_2-\mu_1}{\mu_1}\right)\left(\frac{1}{R_2}-\frac{1}{R_1}\right)$ gives the power of the lens or the reciprocal of the focal length with a negative sign.

$$\therefore \quad -\frac{1}{f} = \frac{\mu_2-\mu_1}{\mu_1}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)$$

$$\therefore \quad \frac{1}{f} = \left(\frac{\mu_2}{\mu_1}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

If $\mu_1 = 1$ and $\mu_2 = \mu$. we get,

$$\boxed{\frac{1}{f} = (\mu-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)} \rightarrow \text{Lens maker formula}$$

R_1 = positive and R_2 = negative by sign convention,

So, $$S_{m(thin)} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

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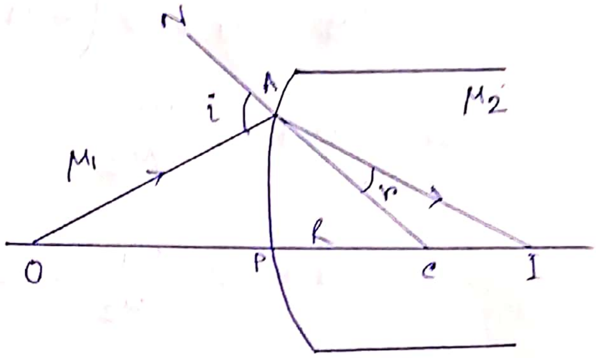
Prove the relation $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$ for refraction at Convex Spherical Surface

Consider, a convex spherical surface of radius of curvature R seperating the two media of refractive index μ_1 and μ_2 $(\mu_2 > \mu_1)$. Consider a ray starting from O incident on the surface at A making an angle i with the normal CAN. It is refracted along AI making an angle of refraction r with the normal.



If $P = -\frac{1}{R}\left(\frac{\mu_2}{\mu_1} - 1\right)$ is the power of the refracting surface and u the distance of the object at O, v is the distance of the image I from pole P, then from O to A the ray undergoes translation in a medium of refractive index μ_1, suffers refraction at A from a medium of refractive index μ_1 to a medium of r.I. μ_2 and undergoes translation from A to I in the medium with r.I. μ_2. The system matrix for the system is,

$$S_m = T_{m2} R_m T_{m1}$$

or, $$S_m = \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -P & \frac{\mu_1}{\mu_2} \end{bmatrix} \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1+0 & u+0 \\ -P+0 & -uP + \frac{\mu_1}{\mu_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 - Pv & u + v\left[-uP + \frac{\mu_1}{\mu_2}\right] \\ 0 - P & 0 + \left[-uP + \frac{\mu_1}{\mu_2}\right] \end{bmatrix}$$

or, $$\begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 - Pv & u + v\left[-uP + \frac{\mu_1}{\mu_2}\right] \\ -P & -uP + \frac{\mu_1}{\mu_2} \end{bmatrix} \begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix}$$

— is the matrix equation for the System.

For paraxial rays emanating from O, $y_1 = 0$. the image plane is, therefore, determined for the condition $y_2 = 0$

but, $y_2 = (1 - Pv)\, y_1 + \left[u + v\left(-uP + \frac{\mu_1}{\mu_2}\right)\right]\alpha_1$

As, $y_1 = 0$, so,

$$u + v\left(-uP + \frac{\mu_1}{\mu_2}\right) = 0$$

or, $$u + v\frac{\mu_1}{\mu_2} - uvP = 0$$

$$\therefore\ u + v\frac{\mu_1}{\mu_2} - uv\frac{\mu_2 - \mu_1}{\mu_2 R} = 0$$

$$\therefore\ u\mu_2 + v\mu_1 = uv\frac{\mu_2 - \mu_1}{R}$$

Dividing both sides by 'uv' we get,

$$\frac{\mu_2}{v} + \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$$

As u is negative and both v and R are positive

$$\boxed{\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}}$$

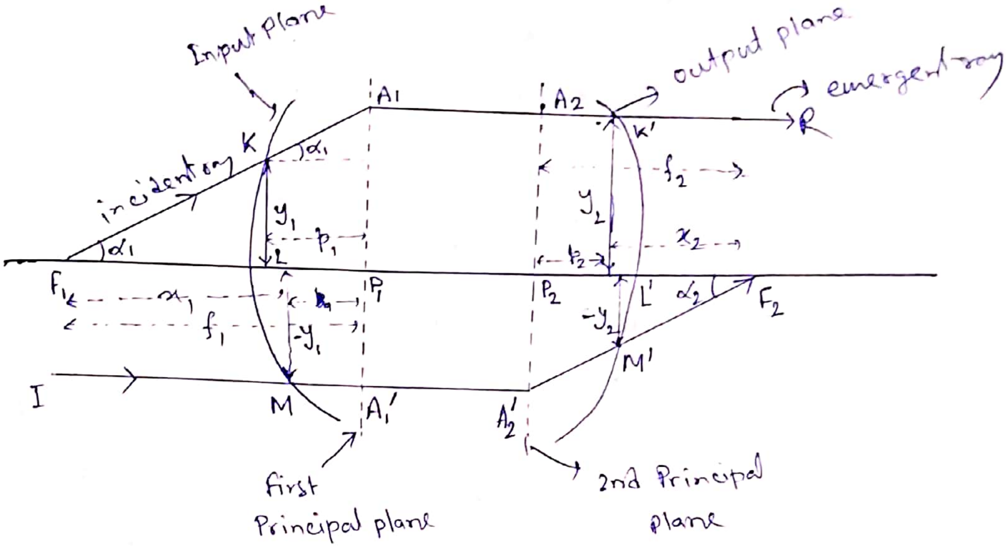
Cardinal Points of lenses.

There are three pairs or six cardinal points. These are

(i) First and second focal points

(ii) first and second principal points (or unit ~~plane~~ points)

(iii) First and second nodal points



(i) Location of focal points :-

Let, (y_1, α_1) be the co-ordinates of the incident ray F_1K at K and (y_2, α_2) of the emergent ray at K', then the system matrix is,

$$S_m = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and matrix equation,

$$\begin{bmatrix} y_2 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ \alpha_1 \end{bmatrix}$$

which gives the ray equations as,

$$y_2 = a y_1 + b\alpha_1 \quad \text{---} (i)$$

$$\alpha_2 = c y_1 + d\alpha_1 \quad \text{---} (ii)$$

first focal point : for the emergent ray K'R, $\alpha_2 = 0$ as it is parallel to the principal axis

From eq^n (ii)

$$0 = c y_1 + d\alpha_1$$

$$\therefore c y_1 = -d\alpha_1$$

$$\therefore y_1/\alpha_1 = -d/c \quad \text{----} (iii)$$

But $\alpha_1 = \tan\alpha_1$ (α_1 being small) $= +\frac{y_1}{-x_1}$ because x_1 is negative

$$\therefore \frac{y_1}{\alpha_1} = -x_1 \quad \text{....} (iv)$$

Comparing (iii) & (iv) we have.

$$x_1 = d/c \quad \text{----- (v)}$$

This gives the distance of the first focal point f_1 from the input plane.

Also. $\tan\alpha_1 = \alpha_1 = \dfrac{A_1 P_1}{f_1 P_1}$

But, $A_1 P_1 = A_2 P_2 = K'L' = y_2$

and $F_1 P_1 = -f_1$ [f_1 is negative by sign convention]

$\therefore \alpha_1 = \dfrac{y_2}{-f_1}$ or $f_1 = -\dfrac{y_2}{\alpha_1} = -\left[\dfrac{+ay_1 + b\alpha_1}{\alpha_1}\right]$

— by eqn (1)

$$\therefore f_1 = -\frac{ay_1}{\alpha_1} - b$$

$$= \frac{ad}{c} - b = \frac{ad - bc}{c} \quad \left[\text{as } \frac{y_1}{\alpha_1} = -\frac{d}{c}\right]$$

Now, $(ad-bc)$ is the value of det $S_m \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and is equal to $\frac{\mu_i}{\mu_o}$ where μ_i is the refractive index of the medium from which light is incident and μ_o that of the medium into which the ray emerges out of the optical system.

$$\therefore \boxed{f_1 = \frac{\mu_i}{\mu_o}\frac{1}{c}} \quad \text{I - (vi)}$$

→ This gives the distance of the first focal ~~plane~~ point from the first principal plane $A_1 P_1 A_1'$ or the first of the optical system.

Second focal Point :-

When a ray like IM parallel to the principal axis is incident on the optical system, after refraction it meets the principal axis at F_2, the second principal focal point. for this ray $\alpha_1 = 0$, the input height is $-y_1$, the output height is $-y_2$ as these heights are below the principal axis and α_2 is negative being measured in the downward direction. Therefore, equation (I) and (II) becomes

$$-y_2 = -a y_1 \quad \text{--- (vii)}$$

$$-\alpha_2 = -c y_1 \quad \text{--- (viii)}$$

$$\therefore \quad \frac{y_2}{\alpha_2} = \frac{a}{c} \quad \text{---} \quad (ix)$$

But, $\alpha_2 = \tan\alpha_2$ (α_2 = small) $= -\frac{y_2}{x_2} = -\frac{y_1}{f_2} \quad \text{---} \quad (x)$

$$\therefore \quad \frac{y_2}{\alpha_2} = -\frac{y_2 x_2}{y_2} = -x_2$$

$$\text{or,} \quad x_2 = -\frac{y_2}{\alpha_2} = -\frac{a}{c} \quad \text{---} \quad (xi)$$

This gives the distance of the 2nd focal point from the output plane of the optical system.

From (x), $\quad f_2 = -\frac{y_1}{\alpha_2}$

$$\therefore \quad f_2 = -\frac{1}{a} \cdot \frac{y_2}{\alpha_2} = -\frac{1}{a} \cdot \frac{a}{c} \quad \left[\text{As } \frac{y_2}{\alpha_2} = \frac{a}{c} \text{ by eq}^n \text{ (ix)}\right]$$

$$\therefore \quad \boxed{f_2 = -\frac{1}{c}} \quad \text{---} \quad (xii)$$

This gives the distance of the second focal point f_2 from the 2nd principal plane $A_2 P_2 A_2'$ or the second focal length of the optical system.

(ii) <u>Location of Principal points</u> :-

In figure above $A_1 P_1 A_1'$ is the first principal plane and the point P_1 where it meets the principal axis is the 'first principal point'.

<u>first principal point</u> :- The distance of the first principal point $P_1 = LP_1 = p_1 = F_1P_1 - F_1L = -f_1 - (-x_1)$

$$\therefore \quad p_1 = x_1 - f_1$$

But, $x_1 = \frac{d}{c}$ [eq^n (v)] and $f_1 = \left(\frac{\mu_i}{\mu_o}\right)\frac{1}{c}$ [eq^n (vi)]

$$\therefore \quad p_1 = \frac{d}{c} - \left(\frac{\mu_i}{\mu_o}\right) \cdot \frac{1}{c}$$

$$\text{or,} \quad \boxed{p_1 = \frac{d - \frac{\mu_i}{\mu_o}}{c}} \quad \text{---} \quad (xiii)$$

Second principal point

$A_2 P_2 A_2'$ is the second principal plane and the point P_2 where it meets the principal axis is the second principal point.

The distance of the second principal point $P_2 = P_2 L' = -p_2$ from the output plane.

$$\therefore -p_2 = P_2 F_2 - L' F_2 = f_2 - x_2$$

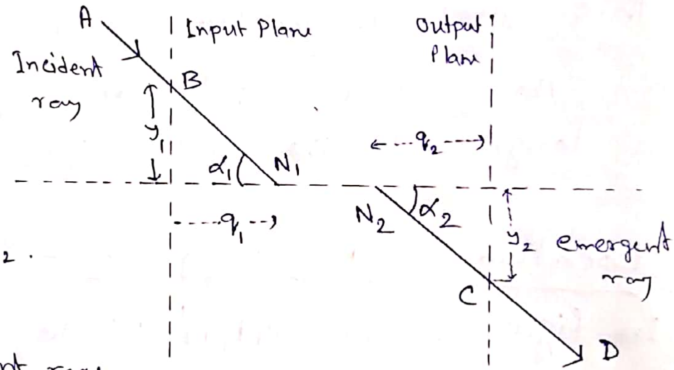
$$\text{or,} \quad p_2 = -f_2 + x_2 = -\left(-\frac{1}{c}\right) + \left(-\frac{a}{c}\right) = \frac{1-a}{c} \quad \text{--- (xiv)}$$

[by eqn (xii) & (xi)]

(iii) Location of nodal points :-

Let N_1 and N_2 be the two nodal points. According to the defination of nodal points, the incident and the emergent rays are parallel. The incident ray AB is directed towards the first nodal point N_1

After refraction through the optical system it comes out along the emergent ray CD as if emerging from the second nodal point N_2.



As the incident and emergent rays are parallel

$$\alpha_1 = \alpha_2 = \alpha$$

Eq's (I) and (II) can be written as

$$y_2 = a y_1 + b\alpha \quad \text{--- (xv)}$$

$$\alpha_2 = \alpha = c y_1 + d\alpha \quad \text{--- (xvi)}$$

First nodal point :- In ΔPBN_1

$$-\alpha = \frac{y_1}{q_1} \qquad [\alpha = \text{small}, \tan\alpha = \alpha]$$

$$\therefore q_1 = -\frac{y_1}{\alpha} \quad \text{--- (xvii)}$$

from eqn (xvi), $\alpha(1-d) = c y_1$

$$\therefore \alpha = \frac{cy_1}{1-d} \quad \text{---} \quad (xviii)$$

Substituting in eq^n (XVII) we get,

$$q_1 = -y_1 \times \frac{1-d}{cy_1} = \frac{d-1}{c} \quad \text{---} \quad (xix)$$

This equation gives the distance of the first nodal point from the input plane.

<u>Second nodal plane</u> : In $\Delta\, QCN_2$.

$$-\alpha = \frac{-y_2}{-q_2} \quad [\text{by sign convention}]$$

$$\text{or, } q_2 = -\frac{y_2}{\alpha} = -\left[\frac{ay_1 + b\alpha}{\alpha}\right] = -\left[\frac{ay_1}{\alpha} + b\right]$$

substituting $\alpha = \dfrac{cy_1}{1-d}$ from eq^n (xviii) we get,

$$q_2 = -\left[\frac{ay_1(1-d)}{cy_1} + b\right]$$

$$= -\left[\frac{a(1-d)}{c} + b\right]$$

$$= \frac{(ad-bc)-a}{c}$$

$$= \frac{\det S_m - a}{c} = \frac{\mu_i/\mu_0 - a}{c}$$

$$\therefore \boxed{q_2 = \frac{(\mu_i/\mu_0) - a}{c}}$$

This equation gives the distance of the second nodal point from the output plane.

Seperation of principal points:

$$p_1 - p_2 = \frac{d - \mu_i/\mu_0}{c} - \frac{1-a}{c} = \frac{d-1-\mu_i/\mu_0 + a}{c}$$

Seperation of nodal points:

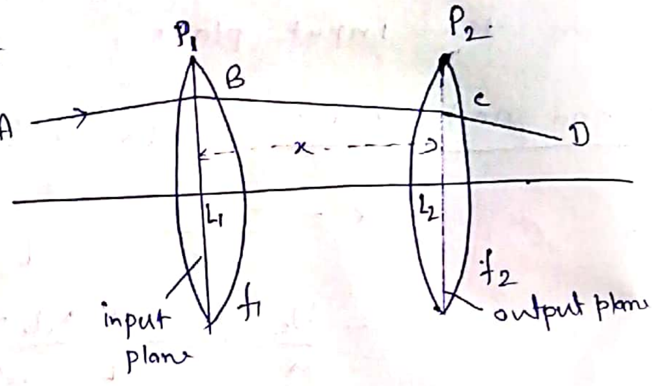
$$q_1 - q_2 = \frac{d-1}{c} - \frac{\mu_i/\mu_0 - a}{c}$$

$$(q_1 - q_2) = \frac{d-1-\mu_i/\mu_0 - a}{c}$$

Combination of two lenses :-

Consider two thin lenses L_1 and L_2 placed co-oxially, seperated by a distance x. Transverse planes P_1Q_1 and P_2Q_2 passing through the center of L_1 and L_2 serve as input plane and output plane respectively.

An incident ray AB incident at B on lens L_1 (of focal length f_1) after refraction through it meets the lens L_2 (of focal length f_2) at C and finally emerges along CD as emergent ray.



The process consists of three steps,

(i) Thin lens refraction at L_1 represented by thin lens matrix,

$$[S_{thin}]_1 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix}$$

(ii) Translation of the ray from B to C represented by translation matrix,

$$T_m = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

(iii) Thin lens refraction at L_2 represented by thin lens matrix

$$[S_{thin}]_2 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix}$$

Hence the system matrix for the combination of two lenses seperated by a distance x may be written as

$$[S_m] = [S_{thin}]_2 [T_m] [S_{thin}]_1$$

$$= \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix} \begin{bmatrix} 1-\frac{x}{f_1} & x \\ -\frac{1}{f_1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\frac{x}{f_1} & x \\ -\frac{1}{f_2}-\frac{1}{f_1}+\frac{x}{f_1 f_2} & 1-\frac{x}{f_2} \end{bmatrix}$$

This is the system matrix for a combination of two lenses a distance x apart.

If the medium on the two sides is the same i.e. $\mu_i = \mu_o$, the focal length of the system is,

$\frac{1}{F} = -c$ where c is the element of system matrix $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Hence the focal length of the combination,

$$\frac{1}{F} = -\left[-\frac{1}{f_2}-\frac{1}{f_1}+\frac{x}{f_1 f_2}\right]$$

$$= \frac{1}{f_1}+\frac{1}{f_2}-\frac{x}{f_1 f_2}$$

F is called the equivalent focal length,

$$\boxed{F = \frac{f_1 f_2}{f_2+f_1-x}}$$

And (f_2+f_1-x) is called the optical seperation between the two lenses.

Problems

① A plano-convex lens 2.0 cm thick has a radius of curvature 10 cm and refractive index 1.5. Find the position of principal planes.

② Radius of curvature of each surface of a bi-convex lens of thickness 8/3 cm and refractive index 2 is 4 cm. If the lens is placed in air determine (i) system matrix (ii) focal length (iii) position of unit planes.

③ A convex lens of focal length 20 cm is seperated from a concave lens of same focal length by a distance of 10 cm. Find the equivalent focal length and position of cardinal points.